Strong 0-dimensionality in Pointfree Topology

Bernhard Banaschewski

McMaster University, Hamilton, Ontario, Canada
iscoe@math.mcmaster.ca

Abstract

Classically, a Tychonoff space is called strongly 0-dimensional if its Stone-Čech compactification is 0-dimensional, and given the familiar relationship between spaces and frames it is then natural to call a completely regular frame strongly 0-dimensional if its compact completely regular coreflection is 0-dimensional (meaning: is generated by its complemented elements). Indeed, it is then seen immediately that a Tychonoff space is strongly 0-dimensional iff the frame of its open sets is strongly 0-dimensional in the present sense.

This talk will provide an account of various aspects of this notion. Particularly relevant for this will be

–the saturation quotient SM of a compact normal frame M, given by the saturation nucleus s_M on M for which $s_M(a) = \bigvee\{x \in M | x \vee y = e \Rightarrow a \vee y = e\}$.

–the variants of normality expressed by the following conditions: if $a \vee b = e$ (the top) then there exists $c \leq b$ such that $a \vee c = e$ for which

$c \prec b$ (normal) $c \prec\!\prec b$ (completely normal) c complemented (strongly normal)

where $c \prec b$ means $b \vee c^* = e$ (c^* the pseudocomplement of c) and $c \prec\!\prec b$ indicates the existence of an infinite sequence of interpolations

$$c \prec b,\ c \prec d_{11} \prec b,\ c \prec d_{21} \prec d_{11} \prec d_{23} \prec b, \ldots$$

–the cozero part $\mathrm{Coz}L$ of a completely regular frame L consisting of all elements $\mathrm{coz}(\gamma) = \gamma((-,0) \vee (0,-))$ for the real-valued continuous functions γ on L (corresponding to the classical cozero *sets* of a space), and

–a representation of the compact completely regular coreflection of a completely regular frame L as

$$S\mathcal{J}\mathrm{Coz}L \to L,\ I \mapsto \bigvee\{\mathrm{coz}(\gamma) | \mathrm{coz}(\gamma) \in I\}$$

where $\mathcal{J}\mathrm{Coz}L$ is the frame of ideals of the lattice $\mathrm{Coz}L$.

The latter will be used to obtain the following pointfree form of a classical result: A completely regular frame L is strongly 0-dimensional iff every cozero elements of L is a countable join of complemented ones.

Further, it will be shown that some familiar types of completely regular frames are strongly 0-dimensional and characterized by very natural additional conditions. This will include the P-frames, that is, the completely regular frames all whose cozero elements are complemented; they turn out to be the strongly 0-dimensional frames in which any countable join of complemented elements is complemented. The latter are called the P_0-frames, and it will be shown that they are reflective in the category of all 0-dimensional frames. This parallels the corresponding result for P-frames in relation to all completely regular frames, but with a substantially simpler proof. It remains a challenging open question whether $P_0 = P$.