Three-Valued Logics with Subclassical Negation

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December 28, 2020

THREE-VALUED LOGICS WITH SUBCLASSICAL NEGATION

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Abstract. We first prove that any [non-classical] three-valued logic with subclassical negation (3VLSN) is defined by a [unique (up to isomorphism)] *superclassical* three-valued matrix (viz., that whose negation reduct has a classical submatrix) and then provide effective algebraic criteria of any 3VLSN's being [sub]classical|having no consistent non-subclassical extension|having a proper paraconsistent/inferentially paracomplete extension. As a by product, we also prove that any implicative/disjunctive paraconsistent/paracomplete 3VLSN has no proper axiomatic consistent non-classical extension, any classical extension being relatively axiomatized by the *Ex Contradictione Quodlibet/Excluded Middled Law* axiom. Likewise, we prove that any [disjunctive non-]classical [(in particular, paraconsistent/paracomplete)] 3VLSN has no proper inferentially consistent [non-classical disjunctive] extension [any classical extension being disjunctive (and relatively axiomatized by the *Resolution* rule/the *Excluded Middled Law* axiom)].

1. Introduction

Perhaps, the principal value of *universal* logical investigations consists in discovering uniform transparent points behind particular results, originally proved *ad hoc*.

On the other hand, appearance of any non-classical (in particular, many-valued) logic inevitably raises the problems of studying both the logic itself and those related to it (including its extensions). In particular, their connections with classical (two-valued) logics deserves a particular emphasis. First of all, this concerns the property of a non-classical logic's being *subclassical* in the sense of being a sublogic of a classical logic, because any classical logic is maximal, that is, has no proper consistent extension. It is then equally valuable to explore whether a given subclassical logic has a consistent non-subclassical extension.

Likewise, when dealing with three-valued logics, in which case a third truth value is invoked to represent incomplete/inconsistent information instead of certain truth and falsehood, as in the classical logic, and so logics become *paracomplete/paraconsistent* (viz., refuting the *Excluded Middle Law* axiom/the *Ex Contradictione Quodlibet* rule), the issue of their *maximal* paracompleteness/paraconsistency in the sense of absence of any proper paracomplete/paraconsistent extension becomes especially acute. Such strong version of maximal paraconsistency — as opposed to the weak *axiomatic* one (regarding merely *axiomatic* extensions) discovered in [18] for P^1 — was first observed in [11] for the *logic of paradox* LP [8] and then for HZ [3] in [14] as well as for arbitrary three-valued expansions of both HZ and the *logic of antinomies* LA [1] in [17], and has been proved for arbitrary conjunctive subclassical three-valued paraconsistent logics in the reference [Pyn 95b] of [11]. In this paper, we provide an effective — in case of finitely many connectives — algebraic criterion of the maximal paraconsistency/inferential

2020 *Mathematics Subject Classification.* 03B20, 03B22, 03B50, 03B53.
Key words and phrases. logic; calculus; matrix; extension.

paracompleteness of three-valued paraconsistent/paracomplete logics with subclassical negation [fragment] properly inherited by their *three-valued* expansions, while any such logic is *axiomatically* maximally paraconsistent/inferentially paracomplete. As a consequence, we prove that any conjunctive/both subclassical and disjunctive/refuting the *Double Negation Law* three-valued paraconsistent logic with subclassical negation is maximally paraconsistent. In particular, any *three-valued* expansion of $LP/HZ/P^1$ is maximally paraconsistent.

2. Basic issues

Notations like img, dom, ker, hom, π_i and Con and related notions are supposed to be clear.

2.1. **Set-theoretical background.**

We follow the standard set-theoretical convention, according to which natural numbers (including 0) are treated as finite ordinals (viz., sets of lesser natural numbers), the ordinal of all them being denoted by ω. Then, given any $(N \cup \{n\}) \subseteq \omega$, set $(N \div n) \triangleq \{\frac{m}{n} \mid m \in N\}$. The proper class of all ordinals is denoted by ∞. Also, functions are viewed as binary relations, while singletons are identified with their unique elements, unless any confusion is possible.

A function f is said to be *singular*, provided $|\operatorname{img} f| \in 2$, that is, $(\ker f) = (\operatorname{dom} f)^2$.

Given a set S, the set of all subsets of S [of cardinality $\in K \subseteq \infty$] is denoted by $\wp_{[K]}(S)$. Then, an *enumeration of* S is any bijection from $|S|$ onto S. As usual, given any equivalence relation θ on S, by ν_θ we denote the function with domain S defined by $\nu_\theta(a) \triangleq \theta[\{a\}]$, for all $a \in S$, whereas we set $(T/\theta) \triangleq \nu_\theta[T]$, for every $T \subseteq S$. Next, S-tuples (viz., functions with domain S) are often written in the either sequence $\bar{t}$ or vector $\vec{t}$ forms, its s-th component (viz., the value under argument s), where $s \in S$, being written as t_s or t^s. Given two more sets A and B, any relation $R \subseteq (A \times B)$ (in particular, a mapping $R : A \to B$) determines the equally-denoted relation $R \subseteq (A^S \times B^S)$ (resp., mapping $R : A^S \to B^S$) pointwise. Likewise, given a set A, an S-tuple $\overline{B}$ of sets and any $\bar{f} \in (\prod_{s \in S} B_s^A)$, put $(\prod \bar{f}) : A \to (\prod \overline{B}), a \mapsto \langle f_s(a)\rangle_{s \in S}$. (In case $I = 2$, $f_0 \times f_1$ stands for $(\prod \bar{f})$.) Further, set $\Delta_S \triangleq \{\langle a, a\rangle \mid a \in S\}$, functions of such a kind being referred to as *diagonal*, and $S^+ \triangleq \bigcup_{i \in (\omega \setminus 1)} S^i$, elements of $S^* \triangleq (S^0 \cup S^+)$ being identified with ordinary finite tuples/sequences, the binary concatenation operation on which being denoted by $*$, as usual. Then, any binary operation $\diamond$ on S determines the equally-denoted mapping $\diamond : S^+ \to S$ as follows: by induction on the length $l = (\operatorname{dom} \bar{a})$ of any $\bar{a} \in S^+$, put:

$$\diamond\bar{a} \triangleq \begin{cases} a_0 & \text{if } l = 1, \\ (\diamond(\bar{a} \restriction (l-1))) \diamond a_{l-1} & \text{otherwise.} \end{cases}$$

In particular, given any $f : S \to S$ and any $n \in \omega$, set $f^n \triangleq (\circ\langle n \times \{f\}, \Delta_S\rangle) : S \to S$. Finally, given any $T \subseteq S$, we have the *characteristic function* $\chi_S^T \triangleq ((T \times \{1\}) \cup ((S \setminus T) \times \{0\}))$ *of* T *in* S.

2.2. **Algebraic background.**

Unless otherwise specified, abstract algebras are denoted by Fraktur letters [possibly, with indices], their carriers being denoted by corresponding Italic letters [with same indices, if any].

Given a Σ-algebra $\mathfrak{A}$, $\mathrm{Con}(\mathfrak{A})$ is a closure system forming a bounded lattice with meet $\theta \cap \vartheta$ of any $\theta, \theta \in \mathrm{Con}(\mathfrak{A})$, their join $\theta \vee \vartheta$, being the transitive closure of $\theta \cup \vartheta$, zero Δ_A and unit A^2. Then, given a class K of Σ-algebras, set $\hom(\mathfrak{A}, \mathsf{K}) \triangleq$

$(\bigcup\{\hom(\mathfrak{A},\mathfrak{B}) \mid \mathfrak{B} \in \mathsf{K}\})$, in which case $\ker[\hom(\mathfrak{A},\mathsf{K})] \subseteq \mathrm{Con}(\mathfrak{A})$, and so $(A^2 \cap \bigcap \ker[\hom(\mathfrak{A},\mathsf{K})]) \in \mathrm{Con}(\mathfrak{A})$.

A *(propositional/sentential) language/signature* is any algebraic (viz., functional) signature Σ (to be dealt with throughout the paper by default) constituted by function (viz., operation) symbols of finite arity to be treated as *(propositional/sentential) connectives.* Given any $\alpha \in \wp_{\infty\setminus 1}(\omega)$, put $V_\alpha \triangleq \{x_\beta \mid \beta \in \alpha\}$, elements of which being viewed as *(propositional/sentential) variables of rank* α, and $(\forall_\alpha) \triangleq (\forall V_\alpha)$. Then, we have the absolutely-free Σ-algebra $\mathfrak{Fm}^\alpha_\Sigma$ freely-generated by the set V_α, its endomorphisms/elements of its carrier $\mathrm{Fm}^\alpha_\Sigma$ being called *(propositional/sentential) Σ-substitutions/-formulas of rank α.* Recall that

$$\forall h \in \hom(\mathfrak{A},\mathfrak{B}) : [(\mathrm{img}\, h) = B) \Rightarrow] \\ (\hom(\mathfrak{Fm}^\alpha_\Sigma,\mathfrak{B}) \supseteq [=]\{h \circ g \mid g \in \hom(\mathfrak{Fm}^\alpha_\Sigma,\mathfrak{A})\}), \quad (2.1)$$

where $\mathfrak{A}$ and $\mathfrak{B}$ are Σ-algebras. Likewise, any $\langle \phi,\psi\rangle \in \mathrm{Eq}^\alpha_\Sigma \triangleq (\mathrm{Fm}^\alpha_\Sigma)^2$ is referred to as a *Σ-equation/indentity of rank α* and normally written in the standard equational form $\phi \approx \psi$. (In general, any mention of α is normally omitted, whenever $\alpha = \omega$.) In this way, given any $h \in \hom(\mathfrak{Fm}^\alpha_\Sigma,\mathfrak{A})$, $\ker h$ is the set of all Σ-identities of rank α *true/satisfied in* $\mathfrak{A}$ *under* h. Likewise, given a class K of Σ-algebras, $\theta^\alpha_\mathsf{K} \triangleq (\mathrm{Eq}^\alpha_\Sigma \cap \bigcap \ker[\hom(\mathfrak{Fm}^\alpha_\Sigma,\mathsf{K})]) \in \mathrm{Con}(\mathfrak{Fm}^\alpha_\Sigma)$ is the set of all all Σ-identities of rank α *true/satisfied in* K, in which case we set $\mathfrak{F}^\alpha_\mathsf{K} \triangleq (\mathfrak{Fm}^\alpha_\Sigma/\theta^\alpha_\mathsf{K})$. (In case both α as well as both K and all members of it are finite, the set $I \triangleq \{\langle h,\mathfrak{A}\rangle \mid h \in \hom(\mathfrak{Fm}^\alpha_\Sigma,\mathfrak{A}), \mathfrak{A} \in \mathsf{K}\}$ is finite — more precisely, $|I| = \sum_{\mathfrak{A}\in\mathsf{K}} |A|^\alpha$, in which case $g \triangleq (\prod_{i\in I}\pi_0(i)) \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \prod_{i\in I}(\pi_1(i){\restriction}\,\mathrm{img}\,\pi_0(i)))$ with $(\ker g) = \theta \triangleq \theta^\alpha_\mathsf{K}$, and so, by the Homomorphism Theorem, $e \triangleq (g \circ \nu_\theta^{-1})$ is an isomorphism from $\mathfrak{F}^\alpha_\mathsf{K}$ onto the subdirect product $(\prod_{i\in I}(\pi_1(i){\restriction}\,\mathrm{img}\,\pi_0(i))){\restriction}(\mathrm{img}\, g)$ of $\langle \pi_1(i){\restriction}\,\mathrm{img}\,\pi_0(i)\rangle_{i\in I}$. In this way, the former is finite, for the latter is so — more precisely, $|F^\alpha_\mathsf{K}| \leqslant (\max_{\mathfrak{A}\in\mathsf{K}} |A|)^{|I|}$.)

The class of all Σ-algebras satisfying every element of an $\mathfrak{I} \subseteq \mathrm{Eq}^\omega_\Sigma$ is called the *variety axiomatized by* $\mathfrak{I}$. Then, the variety $\mathbf{V}(\mathsf{K})$ axiomatized by θ^ω_K is the least variety including K and is said to be *generated by* K, in which case $\theta^\alpha_{\mathbf{V}(\mathsf{K})} = \theta^\alpha_\mathsf{K}$, and so $\mathfrak{F}^\alpha_{\mathbf{V}(\mathsf{K})} = \mathfrak{F}^\alpha_\mathsf{K}$.

Given a variety V of Σ-algebras, by (2.1), we have $\mathfrak{F}^\alpha_\mathsf{V} \in \mathsf{V}$. And what is more, given any $\mathfrak{A} \in \mathsf{V}$ and any $h \in \hom(\mathfrak{Fm}^\alpha_\Sigma,\mathfrak{A})$, as $\theta \triangleq \theta^\alpha_\mathsf{V} \subseteq (\ker h)$, by the Homomorphism Theorem, $g \triangleq (h \circ \nu_\theta^{-1}) \in \hom(\mathfrak{F}^\alpha_\mathsf{V},\mathfrak{A})$, in which case $h = (g \circ \nu_\theta)$, and so $\mathfrak{F}^\alpha_\mathsf{V}$ is a *free V-algebra with α free generators.*

The mapping $\mathrm{Var} : \mathrm{Fm}^\omega_\Sigma \to \wp_\omega(V_\omega)$ assigning the set of all *actually* occurring variables is defined in the standard recursive manner by induction on construction of Σ-formulas.

Given any $[m,]n \in \omega$, by $\sigma_{[m:]+n}$ we denote the Σ-substitution extending $[x_i/x_{i+n}]_{i\in(\omega[\setminus m])}$.

2.2.1. *Distributive lattices.* Let $\Sigma_{+[,01]} \triangleq \{\wedge,\vee[,\bot,\top]\}$ be the [bounded] lattice signature with binary $\wedge$ (conjunction) and $\vee$ (disjunction) [as well as nullary $\bot$ and $\top$ (falsehood/zero and truth/unit constants, respectively)].

Given any $n \in (\omega \setminus 2)$, by $\mathfrak{D}_{n[,01]}$ we denote the [bounded] distributive lattice given by the chain $n \div (n-1)$.

2.3. **Propositional logics and matrices.** A *[finitary/unary] Σ-rule* is any couple $\langle \Gamma,\varphi\rangle$, where $\Gamma \in \wp_{[\omega/(2\setminus 1)]}(\mathrm{Fm}^\omega_\Sigma)$ and $\varphi \in \mathrm{Fm}^\omega_\Sigma$, normally written in the standard sequent form $\Gamma \vdash \varphi$, φ/any element of Γ being referred to as the/a *conclusion/premise of* it. A *(substitutional) Σ-instance* of it is then any Σ-rule of the form $\sigma(\Gamma \vdash \varphi) \triangleq (\sigma[\Gamma] \vdash \sigma(\varphi))$, where σ is a Σ-substitution. As usual, Σ-rules without premises are called Σ-*axioms* and are identified with their conclusions. A[n] *[axiomatic] (finitary/unary)* Σ-*calculus* is then any set $\mathcal{C}$ of (finitary/unary) Σ-rules [without premises], the set of all Σ-instances of its elements being denoted by $\mathrm{SI}_\Sigma(\mathcal{C})$.

A *(propositional/sentential)* Σ-*logic* (cf., e.g., [5]) is any closure operator C over $\mathrm{Fm}_\Sigma^\omega$ that is *structural* in the sense that $\sigma[C(X)] \subseteq C(\sigma[X])$, for all $X \subseteq \mathrm{Fm}_\Sigma^\omega$ and all $\sigma \in \hom(\mathfrak{Fm}_\Sigma^\omega, \mathfrak{Fm}_\Sigma^\omega)$, in which case we set $\equiv_C^\alpha \triangleq \{\langle\phi,\psi\rangle \in (\mathrm{Fm}_\Sigma^\alpha)^2 \mid C(\phi) = C(\psi)\}$, where $\alpha \in \wp_{\infty\setminus 1}(\omega)$. This is said to be *self-extensional*, whenever $\equiv_C^\omega \in \mathrm{Con}(\mathfrak{Fm}_\Sigma^\omega)$, the variety $\mathrm{IV}(C)$ axiomatized by $\equiv_C^\omega$ being called the *intrinsic variety of* C (cf. [12]). Then, C is said to be *[inferentially] (in)consistent*, if $x_1 \notin (\in) C(\varnothing[\cup\{x_0\}])$ [(in which case $\equiv_C^\omega = \mathrm{Eq}_\Sigma^\omega \in \mathrm{Con}(\mathfrak{Fm}_\Sigma^\omega)$, and so C is self-extensional)], the only inconsistent Σ-logic being denoted by IC. Further, a Σ-rule $\Gamma \to \Phi$ is said to be *satisfied in/by* C, provided $\Phi \in C(\Gamma)$, Σ-axioms satisfied in C being referred to as *theorems of* C. Next, a Σ-logic C' is said to be a *(proper) [K-]extension of* C [where $K \subseteq \infty$], whenever $(C[\upharpoonright\wp_K(\mathrm{Fm}_\Sigma^\omega)]) \subseteq (\subsetneq)(C'[\upharpoonright\wp_K(\mathrm{Fm}_\Sigma^\omega)])$, in which case C is said to be a *(proper) [K-]sublogic of* C'. In that case, a[n axiomatic] Σ-calculus $\mathcal{C}$ is said to *axiomatize* C' *(relatively to* C*)*, if C' is the least Σ-logic (being an extension of C and) satisfying every rule in $\mathcal{C}$ [(in which case it is called an *axiomatic extension of* C, while

$$C'(X) = C(X \cup \mathrm{SI}_\Sigma(\mathcal{C})). \tag{2.2}$$

for all $X \subseteq \mathrm{Fm}_\Sigma^\omega$)]. Then, C is said to be *[inferentially] maximal*, whenever it has no proper [inferentially] consistent extension. Furthermore, we have the finitary sublogic $C_\exists$ of C, defined by $C_\exists(X) \triangleq (\bigcup C[\wp_\omega(X)])$, for all $X \subseteq \mathrm{Fm}_\Sigma^\omega$, called the *finitarization of* C. Then, the extension of any finitary (in particular, diagonal) Σ-logic relatively axiomatized by a finitary Σ-calculus is a sublogic of its own finitarization, in which case it is equal to this, and so is finitary (in particular, the Σ-logic axiomatized by a finitary Σ-calculus is finitary). Further, C is said to be *[weakly]* $\bar\wedge$-*conjunctive*, where $\bar\wedge$ is a (possibly, secondary) binary connective of Σ (tacitly fixed throughout the paper), provided $C(\phi \mathbin{\bar\wedge} \psi)[\supseteq] = C(\{\phi,\psi\})$, where $\phi,\psi \in \mathrm{Fm}_\Sigma^\omega$, in which case any extension of C is so. Likewise, C is said to be *[weakly]* $\underline\vee$-*disjunctive*, where $\underline\vee$ is a (possibly, secondary) binary connective of Σ (tacitly fixed throughout the paper), provided $C(X \cup \{\phi \mathbin{\underline\vee} \psi\})[\subseteq] = (C(X \cup \{\phi\}) \cap C(X \cup \{\psi\}))$, where $(X \cup \{\phi,\psi\}) \subseteq \mathrm{Fm}_\Sigma^\omega$, in which case [any extension of C is so, while] the following rules [but the last one]:

$$\begin{aligned} x_0 &\vdash (x_0 \mathbin{\underline\vee} x_1), \qquad &(2.3)\\ (x_0 \mathbin{\underline\vee} x_1) &\vdash (x_1 \mathbin{\underline\vee} x_0), &(2.4)\\ (x_0 \mathbin{\underline\vee} x_0) &\vdash x_0 &(2.5)\end{aligned}$$

are satisfied in C, and so in its extensions, whereas any axiomatic extension of C is $\underline\vee$-disjunctive, in view of (2.2). Furthermore, C is said to *have Deduction Theorem (DT) with respect to* a (possibly, secondary) binary connective $\sqsupset$ of Σ (tacitly fixed throughout the paper), provided, for all $\phi \in X \subseteq \mathrm{Fm}_\Sigma^\omega$ and all $\psi \in C(X)$, it holds that $(\phi \sqsupset \psi) \in C(X \setminus \{\phi\})$, in which case the following axioms:

$$\begin{aligned} &x_0 \sqsupset x_0, \qquad &(2.6)\\ &x_0 \sqsupset (x_1 \sqsupset x_0) &(2.7)\end{aligned}$$

are satisfied in C. Then, C is said to be *weakly* $\sqsupset$-*implicative*, whenever it has DT with respect to $\sqsupset$ and satisfies the *Modus Ponens* rule:

$$\{x_0, x_0 \sqsupset x_1\} \vdash x_1. \tag{2.8}$$

Likewise, C is said to be $\sqsupset$-*implicative*, whenever it is weakly so as well as satisfies the *Peirce Law* axiom (cf. [7]):

$$(((x_0 \sqsupset x_1) \sqsupset x_0) \sqsupset x_0). \tag{2.9}$$

Next, C is said to *have Property of Weak Contraposition (PWC) with respect to* a unary $\sim \in \Sigma$ (tacitly fixed throughout the paper), provided, for all $\phi \in \mathrm{Fm}^{\omega}_{\Sigma}$ and all $\psi \in C(\phi)$, it holds that $\sim\phi \in C(\sim\psi)$. Then, C is said to be *[(axiomatically) maximally]* $\sim$*-paraconsistent*, provided it does not satisfy the *Ex Contradictione Quodlibet* rule:

$$\{x_0, \sim x_0\} \vdash x_1 \tag{2.10}$$

[and has no proper $\sim$-paraconsistent (axiomatic) extension]. Likewise, C is said to be *({axiomatically} maximally) [inferentially]* $(\veebar, \sim)$*-paracomplete*, whenever $(x_1 \veebar \sim x_1) \notin C(\varnothing[\cup\{x_0\}])$ (and has no proper {axiomatic} [inferentially] $(\veebar, \sim)$-paracomplete extension). In general, by C^{EM} we denote the extension of C relatively axiomatized by the *Excluded Middle Law* axiom:

$$x_0 \veebar \sim x_0. \tag{2.11}$$

Finally, C is said to be *theorem-less/purely-inferential*, whenever it has no theorem. Likewise, C is said to be *[non-]pseudo-axiomatic*, provided $\bigcap_{k\in\omega} C(x_k) \nsubseteq [\subseteq] C(\varnothing)$ [in which case it is $(\veebar, \sim)$-paracomplete/(in)consistent iff it is inferentially so].

Definition 2.1. Given a Σ-logic C, the Σ-logic $C_{+/-0}$, defined by:

$$\begin{array}{rcl} (C_{+/-0}{\restriction}\wp_{\infty\setminus 1}(\mathrm{Fm}^{\omega}_{\Sigma})) & \triangleq & (C{\restriction}\wp_{\infty\setminus 1}(\mathrm{Fm}^{\omega}_{\Sigma})), \\ C_{+/-0}(\varnothing) & \triangleq & (\varnothing/(\bigcap\limits_{k\in\omega} C(x_k))), \end{array}$$

is the greatest/least purely-inferential/non-pseudo-axiomatic sublogic/extension of C called the *purely-inferential/non-pseudo-axiomatic version of* C, respectively. $\square$

Remark 2.2. Clearly, $C \mapsto C_{+/-0}$ are monotonic mappings, forming inverse to one another isomorphisms between the posets of all non-pseudo-axiomatic and purely-inferential Σ-logics, such that $C_{-0+0} \subseteq C$. In particular:

(i) the purely-inferential version of the axiomatic extension of a non-pseudo-axiomatic Σ-logic, relatively-axiomatized by an $\mathcal{A} \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$, is relatively axiomatized by $\{x_0 \vdash \sigma_{+1}(\varphi) \mid \varphi \in \mathcal{A}\}$;
(ii) IC_{+0} is a consistent but not inferentially consistent extension of any purely-inferential Σ-logic, and so an inferentially consistent Σ-logic is maximal iff it is both inferentially maximal and not purely-inferential. $\square$

A *(logical)* Σ*-matrix* (cf. [5]) is any couple of the form $\mathcal{A} = \langle\mathfrak{A}, D^{\mathcal{A}}\rangle$, where $\mathfrak{A}$ is a Σ-algebra, called the *underlying algebra of* $\mathcal{A}$, while $D^{\mathcal{A}} \subseteq A$ is called the *truth predicate of* $\mathcal{A}$, elements of $A[\cap D^{\mathcal{A}}]$ being referred to as *[distinguished] values of* $\mathcal{A}$. (In general, matrices are denoted by Calligraphic letters [possibly, with indices], their underlying algebras being denoted by corresponding Fraktur letters [with same indices, if any].) This is said to be *n-valued/[in]consistent/truth(-non)-empty/truth-|false-singular*, where $n \in \omega$, provided $|A| = n/D^{\mathcal{A}} \neq [=]A/D^{\mathcal{A}} = (\neq)\varnothing/|(D^{\mathcal{A}}|(A\setminus D^{\mathcal{A}}))| \in 2$, respectively. Next, given any $\Sigma' \subseteq \Sigma$, $\mathcal{A}$ is said to be a *(*Σ*-)expansion of* its Σ'*-reduct* $(\mathcal{A}{\restriction}\Sigma') \triangleq \langle\mathfrak{A}{\restriction}\Sigma', D^{\mathcal{A}}\rangle$. (Any notation, being specified for single matrices, is supposed to be extended to classes of matrices member-wise.) Finally, $\mathcal{A}$ is said to be *finite[ly generated]/generated by* a $B \subseteq A$, whenever $\mathfrak{A}$ is so.

Given any $\alpha \in \wp_{\infty\setminus 1}(\omega)$ and any class M of Σ-matrices, we have the closure operator $\mathrm{Cn}^{\alpha}_{\mathsf{M}}$ over $\mathrm{Fm}^{\alpha}_{\Sigma}$ defined by $\mathrm{Cn}^{\alpha}_{\mathsf{M}}(X) \triangleq (\mathrm{Fm}^{\alpha}_{\Sigma} \cap \bigcap\{h^{-1}[D^{\mathcal{A}}] \supseteq X \mid \mathcal{A} \in$

$\mathsf{M}, h \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{A})\}$, for all $X \subseteq \mathrm{Fm}^\alpha_\Sigma$, in which case:

$$\mathrm{Cn}^\alpha_{\mathsf{M}}(X) = (\mathrm{Fm}^\alpha_\Sigma \cap \mathrm{Cn}^\omega_{\mathsf{M}}(X)). \tag{2.12}$$

Then, by (2.1), $\mathrm{Cn}^\omega_{\mathsf{M}}$ is a Σ-logic, called the *logic of* M, a Σ-logic C being said to be *[finitely-]defined by* M, provided $C(X) = \mathrm{Cn}_{\mathsf{M}}(X)$, for all $X \in \wp_{[\omega]}(\mathrm{Fm}_\Sigma)$. A Σ-logic is said to be *n-valued*, where $n \in \omega$, whenever it is defined by an n-valued Σ-matrix, in which case it is finitary (cf. [5]), and so is the logic of any finite class of finite Σ-matrices.

As usual, Σ-matrices are treated as first-order model structures of the first-order signature $\Sigma \cup \{D\}$ with unary predicate D, any Σ-rule $\Gamma \vdash \phi$ being viewed as (the universal closure of — depending upon the context) the infinitary equality-free basic strict Horn formula $(\bigwedge \Gamma) \to \phi$ under the standard identification of any propositional Σ-formula ψ with the first-order atomic formula $D(\psi)$.

Remark 2.3. Since any Σ-formula contains just finitely many variables, and so there is a variable not occurring in it, the logic of any class of truth-non-empty Σ-matrices is non-pseudo-axiomatic. □

Remark 2.4. Since any rule with[out] premises is [not] true in any truth-empty matrix, taking Remark 2.3 into account, given any class M of Σ-matrices, the purely-inferential/non-pseudo-axiomatic version of the logic of M is defined by $\mathsf{M} \cup / \setminus \mathsf{S}$, where S is any non-empty class of truth-empty Σ-matrices/resp., the class of all truth-empty members of M. □

Let $\mathcal{A}$ and $\mathcal{B}$ be two Σ-matrices. A *(strict) [surjective] {matrix} homomorphism from $\mathcal{A}$ [on]to $\mathcal{B}$* is any $h \in \hom(\mathfrak{A}, \mathfrak{B})$ such that $[h[A] = B$ and$]$ $D^{\mathcal{A}} \subseteq (=)h^{-1}[D^{\mathcal{B}}]$ ([in which case $\mathcal{B}/\mathcal{A}$ is said to be a *strict surjective {matrix} homomorphic image/counter-image of* $\mathcal{A}/\mathcal{B}$, respectively]), the set of all them being denoted by $\hom^{[\mathrm{S}]}_{(\mathrm{S})}(\mathcal{A}, \mathcal{B})$. Then, by (2.1), we have:

$$(\exists h \in \hom^{[\mathrm{S}]}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})) \Rightarrow (\mathrm{Cn}^\alpha_{\mathcal{B}} \subseteq [=] \, \mathrm{Cn}^\alpha_{\mathcal{A}}), \tag{2.13}$$

$$(\exists h \in \hom^{\mathrm{S}}(\mathcal{A}, \mathcal{B})) \Rightarrow (\mathrm{Cn}^\alpha_{\mathcal{A}}(\varnothing) \subseteq \mathrm{Cn}^\alpha_{\mathcal{B}}(\varnothing)), \tag{2.14}$$

Further, $\mathcal{A}[\neq \mathcal{B}]$ is said to be a *[proper] submatrix of* $\mathcal{B}$, whenever $\Delta_A \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$, in which case we set $(\mathcal{B} \restriction A) \triangleq \mathcal{A}$. Injective/bijective strict homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ are referred to as *embeddings/isomorphisms of/from $\mathcal{A}$ into/onto $\mathcal{B}$*, in case of existence of which $\mathcal{A}$ is said to be *embeddable/isomorphic into/to* $\mathcal{B}$.

Given a Σ-matrix $\mathcal{A}$, $\chi^{\mathcal{A}} \triangleq \chi^{D^{\mathcal{A}}}_A$ is referred to as the *characteristic function of* $\mathcal{A}$. Then, any $\theta \in \mathrm{Con}(\mathfrak{A})$ such that $\theta \subseteq \theta^{\mathcal{A}} \triangleq (\ker \chi^{\mathcal{A}})$, in which case ν_θ is a strict surjective homomorphism from $\mathcal{A}$ onto $(\mathcal{A}/\theta) \triangleq \langle \mathfrak{A}/\theta, D^{\mathcal{A}}/\theta \rangle$, is called a *congruence of* $\mathcal{A}$, the set of all them being denoted by $\mathrm{Con}(\mathcal{A})$. Given any $\theta, \vartheta \in \mathrm{Con}(\mathcal{A})$, the transitive closure $\theta \vee \vartheta$ of $\theta \cup \vartheta$, being a congruence of $\mathfrak{A}$, is then that of $\mathcal{A}$, for $\theta^{\mathcal{A}}$, being an equivalence relation, is transitive. In particular, any maximal congruence of $\mathcal{A}$ (that exists, by Zorn's Lemma, because $\mathrm{Con}(\mathcal{A}) \ni \Delta_A$ is both non-empty and inductive, for $\mathrm{Con}(\mathfrak{A})$ is so) is the greatest one to be denoted by $\partial(\mathcal{A})$. Finally, $\mathcal{A}$ is said to be *[hereditarily] simple*, provided it has no non-diagonal congruence [and no non-simple submatrix].

Remark 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Σ-matrices and $h \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$. Then, $\theta^{\mathcal{A}} = h^{-1}[\theta^{\mathcal{B}}]$, while $h^{-1}[\theta] \in \mathrm{Con}(\mathfrak{A})$, for all $\theta \in \mathrm{Con}(\mathfrak{B})$. Therefore, $h^{-1}[\theta] \in \mathrm{Con}(\mathcal{A})$, for all $\theta \in \mathrm{Con}(\mathcal{B})$. In particular (when $\theta = \Delta_B$), $(\ker h) \in \mathrm{Con}(\mathcal{A})$, and so h is injective, whenever $\mathcal{A}$ is simple.* □

A Σ-matrix $\mathcal{A}$ is said to be a *[K-]model of* a Σ-logic C [where $K \subseteq \infty$], provided C is a [K-]sublogic of the logic of $\mathcal{A}$ (and $\mathfrak{A} \in \mathsf{K}$), the class of all (simple of) them being denoted by $\mathrm{Mod}^{(*)}_{[K]}(C)$. Next, $\mathcal{A}$ is said to be *$\sim$-paraconsistent/$(\veebar,\sim)$-paracomplete*, whenever the logic of $\mathcal{A}$ is so. Further, $\mathcal{A}$ is said to be *[weakly] $\diamond$-conjunctive*, where $\diamond$ is a (possibly, secondary) binary connective of Σ, provided $(\{a,b\} \subseteq D^{\mathcal{A}})[\Leftarrow] \Leftrightarrow ((a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}})$, for all $a,b \in A$, that is, the logic of $\mathcal{A}$ is [weakly] $\diamond$-conjunctive. Then, $\mathcal{A}$ is said to be *[weakly] $\diamond$-disjunctive*, whenever $\langle \mathfrak{A}, A \setminus D^{\mathcal{A}}\rangle$ is [weakly] $\diamond$-conjunctive, in which case [that is] the logic of $\mathcal{A}$ is [weakly] $\diamond$-disjunctive, and so is the logic of any class of [weakly] $\diamond$-disjunctive Σ-matrices. Likewise, $\mathcal{A}$ is said to be *$\diamond$-implicative*, whenever $((a \in D^{\mathcal{A}}) \Rightarrow (b \in D^{\mathcal{A}})) \Leftrightarrow ((a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}})$, for all $a,b \in A$, in which case it is $\veebar_\diamond$-disjunctive, where $(x_0 \veebar_\diamond x_1) \triangleq ((x_0 \diamond x_1) \diamond x_1)$, while the logic of $\mathcal{A}$ is $\diamond$-implicative, for both (2.8) and $(2.9) = ((x_0 \sqsupset x_1) \veebar_\sqsupset x_0)$ are true in any $\sqsupset$-implicative (and so $\veebar_\sqsupset$-disjunctive) Σ-matrix, while DT is immediate, and so is the logic of any class of $\diamond$-implicative Σ-matrices. Finally, given any (possibly secondary) unary connective $\neg$ of Σ, put $(x_0 \diamond^{\neg} x_1) \triangleq \neg(\neg x_0 \diamond \neg x_1)$. Then, $\mathcal{A}$ is said to be *[weakly] (classically) $\neg$-negative*, provided, for all $a \in A$, $(a \in D^{\mathcal{A}})[\Leftarrow] \Leftrightarrow (\neg^{\mathfrak{A}} a \notin D^{\mathcal{A}})$.

Remark 2.6. Let $\diamond$ and $\neg$ be as above. Then, the following hold:

(i) any $\neg$-negative Σ-matrix:
  a) is [weakly] $\diamond$-disjunctive/-conjunctive iff it is [weakly] $\diamond^{\neg}$-conjunctive/-disjunctive, respectively;
  b) defines a logic having PWC with respect to $\neg \in \Sigma$;

(ii) given any two Σ-matrices $\mathcal{A}$ and $\mathcal{B}$ and any $h \in \hom^{[\mathrm{S}]}_{\mathrm{S}}(\mathcal{A},\mathcal{B})$, $\mathcal{A}$ is (weakly) $\neg$-negative|$\diamond$-conjunctive/-disjunctive/-implicative if[f] $\mathcal{B}$ is so. $\square$

Given a set I and an I-tuple $\overline{\mathcal{A}}$ of Σ-matrices, [any submatrix $\mathcal{B}$ of] the Σ-matrix $(\prod_{i\in I}\mathcal{A}_i) \triangleq \langle \prod_{i\in I}\mathfrak{A}_i, \prod_{i\in I} D^{\mathcal{A}_i}\rangle$ is called the [a] *[sub]direct product of $\overline{\mathcal{A}}$* [whenever, for each $i \in I$, $\pi_i[B] = A_i$]. As usual, when $I = 2$, $\mathcal{A}_0 \times \mathcal{A}_1$ stands for the direct product involved. Likewise, if $(\mathrm{img}\,\overline{\mathcal{A}}) \subseteq \{\mathcal{A}\}$ (and $I = 2$), where $\mathcal{A}$ is a Σ-matrix, $\mathcal{A}^I \triangleq (\prod_{i\in I}\mathcal{A}_i)$ [resp., $\mathcal{B}$] is called the [a] *[sub]direct I-power (square) of $\mathcal{A}$*.

Given a class M of Σ-matrices, the class of all surjective homomorphic [counter-]images/(consistent) {truth-non-empty} submatrices of members of M is denoted by $(\mathbf{H}^{[-1]}/\mathbf{S}^{\{*\}}_{(*)})(\mathsf{M})$, respectively. Likewise, the class of all [sub]direct products of tuples (of cardinality $\in K \subseteq \infty$) constituted by members of M is denoted by $\mathbf{P}^{[\mathrm{SD}]}_{(K)}(\mathsf{M})$.

Lemma 2.7. *Let M be a class of Σ-matrices. Then, $\mathbf{H}(\mathbf{H}^{-1}(\mathsf{M})) \subseteq \mathbf{H}^{-1}(\mathbf{H}(\mathsf{M}))$.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be Σ-matrices, $\mathcal{C} \in \mathsf{M}$ and $(h|g) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B},\mathcal{C}|\mathcal{A})$. Then, by Remark 2.5, $(\ker(h|g)) \in \mathrm{Con}(\mathcal{B})$, in which case $(\ker(h|g)) \subseteq \theta \triangleq ((\ker h)\vee(\ker g)) \in \mathrm{Con}(\mathcal{B})$, and so, by the Homomorphism Theorem, $(\nu_\theta \circ (h|g)^{-1}) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{C}|\mathcal{A}, \mathcal{B}/\theta)$, as required. $\square$

Lemma 2.8 (Finitely-Generated Model Lemma)**.** *Let M be a finite class of finite Σ-matrices and $\mathcal{A}$ a finitely-generated (in particular, finite) [truth-non-empty] consistent model of the logic of M. Then, $\mathcal{A} \in \mathbf{H}(\mathbf{H}^{-1}(\mathbf{P}^{\mathrm{SD}}_{\omega\setminus 1}(\mathbf{S}^{[*]}_{*}(\mathsf{M}))/\mathbf{S}^{[*]}_{*}(\mathsf{M})))/$, provided $\mathcal{A}$ is $\veebar$-disjunctive, while members of M are all weakly $\veebar$-disjunctive*

Proof. Take any $A' \in \wp_{\omega\setminus 1}(A)$ generating $\mathfrak{A}$. In that case, $n \triangleq |A'| \in (\omega \setminus 1)$. Let $h \in \hom(\mathfrak{Fm}^n_\Sigma, \mathfrak{A})$ extend any bijection from V_n onto A', in which case $(\mathrm{img}\,h) = A$, and so h is a strict surjective homomorphism from $\mathcal{D} \triangleq \langle \mathfrak{Fm}^n_\Sigma, T\rangle$ onto $\mathcal{A}$, where $T \triangleq h^{-1}[D^{\mathcal{A}}]$. Then, as $\mathcal{A}$ is consistent, by (2.12), we have $\mathrm{Fm}^n_\Sigma \supsetneq T \supseteq \mathrm{Cn}^n_{\mathcal{A}}(T) \supseteq \mathrm{Cn}^n_{\mathsf{M}}(T) = (\mathrm{Fm}^n_\Sigma \cap \bigcap \mathcal{U})$, where $\mathcal{U} \triangleq \{g^{-1}[D^{\mathcal{B}}] \supseteq T \mid \mathcal{B} \in \mathsf{M}, g \in \hom(\mathfrak{Fm}^n_\Sigma, \mathfrak{B})\}$ is both non-empty, for $T \neq \mathrm{Fm}^n_\Sigma$, and finite, for n as well as both M and all members of it are so [while T is non-empty, for $D^{\mathcal{A}}$ is so]. Consider the respective complementary cases:

- $\mathcal{A}$ is $\veebar$-disjunctive, while members of M are all weakly $\veebar$-disjunctive.
  Let us prove, by contradiction, that $T \in \mathcal{U}$. For suppose $T \notin \mathcal{U}$. Take any bijection $\overline{U} : m \triangleq |\mathcal{U}| \to \mathcal{U}$. Then, for each $i \in m$, we have $T \subsetneq U_i$, in which case $U_i \nsubseteq T$, and so there is some $\varphi_i \in (U_i \setminus T) \neq \varnothing$. In this way, as $m \in (\omega \setminus 1)$, while every member of M is weakly $\veebar$-disjunctive, whereas $\mathcal{A}$ is $\veebar$-disjunctive, we get $(\veebar\bar{\varphi}) \in ((\mathrm{Fm}^n_\Sigma \cap \bigcap \mathcal{U}) \setminus T) = \varnothing$. This contradiction implies that $T \in \mathcal{U}$, in which case there are some $\mathcal{B} \in \mathsf{M}$ and some $g \in \hom(\mathfrak{Fm}^n_\Sigma, \mathfrak{B})$ such that $T = g^{-1}[D^{\mathcal{B}}]$, and so $g \in \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{B})$. Then, $E \triangleq (\mathrm{img}\, g)$ forms a subalgebra of $\mathfrak{B}$, in which case $\mathcal{E} \triangleq (\mathcal{B} {\restriction} (\mathrm{img}\, g)) \in \mathbf{S}(\mathsf{M})$, and so $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{E})$. In particular, $\mathcal{E}$ is consistent [and truth-non-empty], for $\mathcal{D}$ is so. Thus, $\mathcal{E} \in \mathbf{S}^{[*]}_*(\mathsf{M})$.
- otherwise.
  For every $i \in I \triangleq (\mathcal{U} \setminus \{\mathrm{Fm}^n_\Sigma\})$, there are some $\mathcal{B}_i \in \mathsf{M}$ and some $f_i \in \hom(\mathfrak{Fm}^n_\Sigma, \mathfrak{B}_i)$ such that $i = f_i^{-1}[D^{\mathcal{B}_i}]$, in which case $E_i \triangleq (\mathrm{img}\, f_i)$ forms a subalgebra of $\mathfrak{B}_i$, and so $\mathcal{E}_i \triangleq (\mathcal{B}_i {\restriction} E_i) \in \mathbf{S}^{[*]}_*(\mathsf{M})$, for $i \neq \mathrm{Fm}^n_\Sigma$ [and $i \supseteq T \neq \varnothing$ is not empty]. Then, since $\mathrm{Fm}^n_\Sigma \neq T = (\mathrm{Fm}^n_\Sigma \cap \bigcap I)$, $|I| \in (\omega \setminus 1)$, while $g \triangleq (\prod_{i \in I} f_i) \in \hom_{\mathrm{S}}(\mathcal{D}, \prod_{i \in I} \mathcal{E}_i)$, whereas, for each $i \in I$, $(\pi_i \circ g) = f_i$, in which case $\pi_i[\mathrm{img}\, g] = E_i$, and so g is a strict surjective homomorphism from $\mathcal{D}$ onto $\mathcal{E} \triangleq ((\prod_{i \in I} \mathcal{E}_i) {\restriction} (\mathrm{img}\, g)) \in \mathbf{P}^{\mathrm{SD}}_{\omega \setminus 1}(\mathbf{S}^{[*]}_*(\mathsf{M}))$.

Thus, $\mathcal{E} \in (\mathbf{P}^{\mathrm{SD}}_{\omega \setminus 1}(\mathbf{S}^{[*]}_*(\mathsf{M}))/\mathbf{S}^{[*]}_*(\mathsf{M}))$, $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{E})$ and $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{A})$. □

Given any Σ-logic C and any $\Sigma' \subseteq \Sigma$, in which case $\mathrm{Fm}^\alpha_{\Sigma'} \subseteq \mathrm{Fm}^\alpha_\Sigma$ and $\hom(\mathfrak{Fm}^\alpha_{\Sigma'}, \mathfrak{Fm}^\alpha_{\Sigma'}) = \{h {\restriction} \mathrm{Fm}^\alpha_{\Sigma'} \mid h \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{Fm}^\alpha_\Sigma), h[\mathrm{Fm}^\alpha_{\Sigma'}] \subseteq \mathrm{Fm}^\alpha_{\Sigma'}\}$, for all $\alpha \in \wp_{\infty \setminus 1}(\omega)$, we have the Σ'-logic C', defined by $C'(X) \triangleq (\mathrm{Fm}^\omega_{\Sigma'} \cap C(X))$, for all $X \subseteq \mathrm{Fm}^\omega_{\Sigma'}$, called the *$\Sigma'$-fragment of C*, in which case C is said to be a *(Σ-)expansion of C'*. In that case, given also any class M of Σ-matrices defining C, C' is, in its turn, defined by $\mathsf{M} {\restriction} \Sigma'$.

2.3.1. *Classical matrices and logics.* A two-valued consistent Σ-matrix $\mathcal{A}$ is said to be *$\sim$-classical*, whenever it is $\sim$-negative, in which case it is truth-non-empty, for it is consistent, and so is both false- and truth-singular, the unique element of $(A \setminus D^{\mathcal{A}})/D^{\mathcal{A}}$ being denoted by $(0/1)_{\mathcal{A}}$, respectively (the index $_{\mathcal{A}}$ is often omitted, unless any confusion is possible), in which case $A = \{0, 1\}$, while $\sim^{\mathfrak{A}} i = (1 - i)$, for each $i \in 2$, whereas $\theta^{\mathcal{A}}$ is diagonal, for $\chi^{\mathcal{A}}$ is so, and so $\mathcal{A}$ is simple but is not $\sim$-paraconsistent.

A Σ-logic is said to be *$\sim$-[sub]classical*, whenever it is [a sublogic of] the logic of a $\sim$-classical Σ-matrix, in which case it is inferentially consistent. Then, $\sim$ is called a *subclassical negation for* a Σ-logic C, whenever the $\sim$-fragment of C is $\sim$-subclassical, in which case:

$$\sim^m x_0 \notin C(\sim^n x_0), \tag{2.15}$$

for all $m, n \in \omega$ such that the integer $m - n$ is odd.

Lemma 2.9. *Let $\mathcal{A}$ be a $\sim$-classical Σ-matrix, C the logic of $\mathcal{A}$ and $\mathcal{B}$ a finitely-generated truth-non-empty consistent model of C. Then, $\mathcal{A}$ is embeddable into a strict surjective homomorphic image of $\mathcal{B}$. In particular, $\mathcal{A}$ is isomorphic to any $\sim$-classical model of C, and so C has no proper $\sim$-classical extension.*

Proof. Then, by Lemmas 2.7 and 2.8, there are some non-empty set I, some submatrix $\mathcal{D}$ of $\mathcal{A}^I$, some strict surjective homomorphic image $\mathcal{E}$ of $\mathcal{B}$ and some $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{E})$, in which case $\mathcal{D}$ is truth-non-empty, for $\mathcal{B}$ is so, and so $a \triangleq (I \times \{1\}) \in$

D, in which case $D \ni \sim^{\mathfrak{D}} a = (I \times \{0\})$, and so, as $I \neq \varnothing$, $e \triangleq \{\langle b, (I \times \{b\}) \rangle \mid b \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case $(h \circ e) \in \hom_{\mathrm{S}}^{[\mathrm{S}]}(\mathcal{A}, \mathcal{B})$, and so (2.13), Remark 2.5 and the fact that any $\sim$-classical Σ-matrix has no proper submatrix complete the argument. $\square$

In view of Lemma 2.9, any $\sim$-classical Σ-logic is defined by a unique (either up to isomorphism or when dealing with merely *canonical* $\sim$-classical Σ-matrices, i.e., those of the form $\mathcal{A}$ with $A = 2$ and $a_{\mathcal{A}} = a$, for all $a \in A$, in which case isomorphic ones are clearly equal) $\sim$-classical Σ-matrix, the unique canonical one being said to be *characteristic for|of* the logic.

Corollary 2.10. *Any $\sim$-classical Σ-logic is inferentially maximal.*

Proof. Let $\mathcal{A}$ be a $\sim$-classical Σ-matrix, C the logic of $\mathcal{A}$ and C' an inferentially consistent extension of C. Then, $x_1 \notin T \triangleq C(x_0) \ni x_0$. On the other hand, by the structurality of C', $\langle \mathfrak{Fm}_{\Sigma}^{\omega}, T \rangle$ is a model of C' (in particular, of C), and so is its finitely-generated consistent truth-non-empty submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}_{\Sigma}^{2}, T \cap \mathrm{Fm}_{\Sigma}^{2} \rangle$, in view of (2.13). In this way, (2.13) and Lemma 2.9 complete the argument. $\square$

3. Preliminary advanced key generic issues

3.1. **False-singular consistent weakly conjunctive matrices.**

Lemma 3.1. *Let $\bar{\wedge}$ be a (possibly, secondary) binary connective of Σ, $\mathcal{A}$ a false-singular weakly $\bar{\wedge}$-conjunctive Σ-matrix, $f \in (A \setminus D^{\mathcal{A}})$, I a finite set, $\overline{\mathcal{C}}$ an I-tuple constituted by consistent submatrices of $\mathcal{A}$ and $\mathcal{B}$ a subdirect product of $\overline{\mathcal{C}}$. Then, $(I \times \{f\}) \in B$.*

Proof. By induction on the cardinality of any $J \subseteq I$, let us prove that there is some $a \in B$ including $(J \times \{f\})$. First, when $J = \varnothing$, take any $a \in C \neq \varnothing$, in which case $(J \times \{f\}) = \varnothing \subseteq a$. Now, assume $J \neq \varnothing$. Take any $j \in J \subseteq I$, in which case $K \triangleq (J \setminus \{j\}) \subseteq I$, while $|K| < |J|$, and so, as $\mathcal{C}_j$ is a consistent submatrix of the false-singular matrix $\mathcal{A}$, we have $f \in C_j = \pi_j[B]$. Hence, there is some $b \in B$ such that $\pi_j(b) = f$, while, by induction hypothesis, there is some $a \in B$ including $(K \times \{f\})$. Therefore, since $J = (K \cup \{j\})$, while $\mathcal{A}$ is both weakly $\bar{\wedge}$-conjunctive and false-singular, we have $B \ni c \triangleq (a \mathbin{\bar{\wedge}^{\mathfrak{B}}} b) \supseteq (J \times \{f\})$. Thus, when $J = I$, we eventually get $B \ni (I \times \{f\})$, as required. $\square$

3.2. **Congruence and equality determinants.** A *[binary] relational Σ-scheme* is any Σ-calculus $\varepsilon \subseteq (\wp_{\omega}(\mathrm{Fm}_{\Sigma}^{[2\cap]\omega}) \times \mathrm{Fm}_{\Sigma}^{[2\cap]\omega})$, in which case, given any Σ-matrix $\mathcal{A}$, we set $\theta_{\varepsilon}^{\mathcal{A}} \triangleq \{\langle a, b \rangle \in A^2 \mid \mathcal{A} \models (\forall_{\omega\setminus 2} \bigwedge \varepsilon)[x_0/a, x_1/b]\} \subseteq A^2$. Note that, given a one more Σ-matrix $\mathcal{B}$ and an $h \in \hom_{\{\mathrm{S}/\}}^{(\mathrm{S})}(\mathcal{A}, \mathcal{B})/$, while ε is axiomatic, we have:

$$h^{-1}[\theta_{\varepsilon}^{\mathcal{B}}]\{\subseteq /\}(\supseteq)[\supseteq]\theta_{\varepsilon}^{\mathcal{A}}. \tag{3.1}$$

A *[unary] unitary relational Σ-scheme* is any $\Upsilon \subseteq \mathrm{Fm}_{\Sigma}^{[1\cap]\omega}$, in which case we have the [binary] relational Σ-scheme $\varepsilon_{\Upsilon} \triangleq \{(\upsilon[x_0/x_i]) \vdash (\upsilon[x_0/x_{1-i}]) \mid i \in 2, \upsilon \in \sigma_{1:+1}[\Upsilon]\}$ such that $\theta_{\varepsilon_{\Upsilon}}^{\mathcal{A}}$, where $\mathcal{A}$ is any Σ-matrix, is an equivalence relation on A.

A *[binary] congruence/equality determinant for* a class of Σ-matrices M is any [binary] relational Σ-scheme ε such that, for each $\mathcal{A} \in \mathsf{M}$, $\theta_{\varepsilon}^{\mathcal{A}} \in \mathrm{Con}(\mathcal{A})/ = \Delta_A$, respectively.

Then, according to [16]/[15], a *[unary] unitary congruence/equality determinant for* a class of Σ-matrices M is any [unary] unitary relational Σ-scheme Υ such that ε_{Υ} is a/an congruence/equality determinant for M. (It is unary unitary equality determinants that are equality determinants in the sense of [15].)

Lemma 3.2 (cf., e.g., [16])**.** $\mathrm{Fm}^{\omega}_{\Sigma}$ *is a unitary congruence determinant for every Σ-matrix $\mathcal{A}$.*

Proof. We start from proving the fact the equivalence relation $\theta^{\mathcal{A}} \triangleq \theta^{\mathcal{A}}_{\varepsilon_{\mathrm{Fm}^{\omega}_{\Sigma}}} \in \mathrm{Con}(\mathfrak{A})$. For consider any $\varsigma \in \Sigma$ of arity $n \in \omega$, any $i \in n$, in which case $n \neq 0$, any $\vec{a} \in \theta^{\mathcal{A}}$, any $\bar{b} \in A^{n-1}$, any $\phi \in \mathrm{Fm}^{\omega}_{\Sigma}$ and any $\bar{c} \in A^{\omega}$. Put $\psi \triangleq \varsigma(\langle\langle x_{j+1}\rangle_{j\in i}, x_0\rangle * \langle x_k\rangle_{k\in(n\setminus i)})$ and $\varphi \triangleq ((\sigma_{1:+n}\phi)[x_0/\psi]) \in \mathrm{Fm}^{\omega}_{\Sigma}$. Then, we have

$$
\begin{aligned}
&(\sigma_{1:+1}\phi)^{\mathfrak{A}}[x_{l+1}/c_l; x_0/\varsigma^{\mathfrak{A}}(\langle\langle b_j\rangle_{j\in i}, a_0\rangle * \langle b_k\rangle_{k\in((n-1)\setminus i)})]_{l\in\omega} = \\
&\qquad (\sigma_{1:+1}\varphi)^{\mathfrak{A}}[x_{l+n+1}/c_l; x_0/a_0; x_{m+1}/b_m]_{l\in\omega; m\in(n-1)} \in D^{\mathcal{A}} \Leftrightarrow \\
&\qquad D^{\mathcal{A}} \ni (\sigma_{1:+1}\varphi)^{\mathfrak{A}}[x_{l+n+1}/c_l; x_0/a_1; x_{m+1}/b_m]_{l\in\omega; m\in(n-1)} = \\
&\qquad\qquad (\sigma_{1:+1}\phi)^{\mathfrak{A}}[x_{l+1}/c_l; x_0/\varsigma^{\mathfrak{A}}(\langle\langle b_j\rangle_{j\in i}, a_1\rangle * \langle b_k\rangle_{k\in((n-1)\setminus i)})]_{l\in\omega},
\end{aligned}
$$

in which case we eventually get

$$\langle \varsigma^{\mathfrak{A}}(\langle\langle b_j\rangle_{j\in i}, a_0\rangle * \langle b_k\rangle_{k\in((n-1)\setminus i)}), \varsigma^{\mathfrak{A}}(\langle\langle b_j\rangle_{j\in i}, a_1\rangle * \langle b_k\rangle_{k\in((n-1)\setminus i)})\rangle \in \theta^{\mathcal{A}},$$

and so $\theta^{\mathcal{A}} \in \mathrm{Con}(\mathfrak{A})$. Finally, as $x_0 \in \mathrm{Fm}^{\omega}_{\Sigma}$, we clearly have $\theta^{\mathcal{A}}[D^{\mathcal{A}}] \subseteq D^{\mathcal{A}}$, as required. □

Lemma 3.3. *Let $\mathcal{A}$ be a Σ-matrix and ε a congruence determinant for $\mathcal{A}$. Then, $\eth(\mathcal{A}) = \theta^{\mathcal{A}}_{\varepsilon}$. In particular, $\mathcal{A}$ is simple, whenever ε is an equality determinant for it.*

Proof. Consider any $\theta \in \mathrm{Con}(\mathcal{A})$ and any $\langle a, b\rangle \in \theta$. Then, as $\mathrm{Con}(\mathcal{A}) \ni \theta^{\mathcal{A}}_{\varepsilon} \supseteq \Delta_A \ni \langle a, a\rangle$, we have $\mathcal{A} \models (\forall_{\omega\setminus 2} \bigwedge \varepsilon)[x_0/a, x_1/a]$, in which case, by the reflexivity of θ, we get $\mathcal{A} \models (\forall_{\omega\setminus 2} \bigwedge \varepsilon)[x_0/a, x_1/b]$, and so $\langle a, b\rangle \in \theta^{\mathcal{A}}_{\varepsilon}$, as required. □

Lemma 3.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be Σ-matrices, ε a/an congruence/equality determinant for $\mathcal{B}$ and h a/an strict homomorphism/embedding from/of $\mathcal{A}$ to/into $\mathcal{B}$. Suppose either ε is binary or $h[A] = B$. Then, ε is a/an congruence/equality determinant for $\mathcal{A}$.*

Proof. In that case, by (3.1), we have $\theta^{\mathcal{A}}_{\varepsilon} = h^{-1}[\theta^{\mathcal{B}}_{\varepsilon}]$. In this way, Remark 2.5/the injectivity of h completes the argument. □

Corollary 3.5. *Let $\mathcal{A}$ be a Σ-matrix. Then, the following are equivalent:*

(i) *$\mathcal{A}$ is hereditarily simple;*
(ii) *$\mathcal{A}$ has a binary equality determinant;*
(iii) *$\mathcal{A}$ has a unary binary equality determinant.*

Proof. First, (ii) is a particular case of (iii). Next, (ii)⇒(i) is by Lemmas 3.3 and 3.4.

Finally, assume (i) holds. Consider any $a, b \in A$. Let $\mathcal{B}$ be the submatrix of $\mathcal{A}$ generated by $\{a, b\}$. Then, it is simple, by (i). Therefore, by Lemmas 3.2 and 3.3, $\Delta_B = \theta^{\mathcal{B}}_{\varepsilon_{\mathrm{Fm}^{\omega}_{\Sigma}}}$. On the other hand, we have the unary binary relational Σ-scheme $\varepsilon \triangleq (\bigcup\{\sigma[\varepsilon_{\mathrm{Fm}^{\omega}_{\Sigma}}] \mid \sigma \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{Fm}^{2}_{\Sigma}), \sigma(x_{0/1}) = x_{0/1}\})$ such that $(\langle a, b\rangle \in \theta^{\mathcal{B}}_{\varepsilon_{\mathrm{Fm}^{\omega}_{\Sigma}}}) \Leftrightarrow (\langle a, b\rangle \in \theta^{\mathcal{B}}_{\varepsilon})$, for $\mathfrak{B}$ is generated by $\{a, b\}$. In this way, by (3.1) with $h = \Delta_B$, we get $(a = b) \Leftrightarrow (\langle a, b\rangle \in \theta^{\mathcal{B}}_{\varepsilon}) \Leftrightarrow (\langle a, b\rangle \in \theta^{\mathcal{A}}_{\varepsilon})$. Thus, ε is an equality determinant for $\mathcal{A}$, and so (iii) holds, as required. □

Lemma 3.6. *Any axiomatic binary equality determinant ε for a class M of Σ-matrices is so for $\mathbf{P}(\mathsf{M})$.*

Proof. In that case, members of M are models of the infinitary universal strict Horn theory $\varepsilon[x_1/x_0] \cup \{(\bigwedge \varepsilon) \to (x_0 \approx x_1)\}$ with equality, and so are well-known to be those of $\mathbf{P}(\mathsf{M})$, as required. □

3.3. **Self-extensional logics versus simple matrices.**

Lemma 3.7. *Let C be a Σ-logic, $\theta \in \mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$, $\mathcal{A} \in \mathrm{Mod}(C)$ and $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$. Suppose $\theta \subseteq \equiv^{\omega}_{C}$. Then, $h[\theta] \subseteq \eth(\mathcal{A})$.*

Proof. Consider any $\langle \phi, \psi \rangle \in \theta$, any $g \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ such that $g(x_{0/1}) = h(\phi/\psi)$ and any $\varphi \in \mathrm{Fm}^{\omega}_{\Sigma}$. Then, $V \triangleq (\mathrm{Var}(\sigma_{1:+1}(\varphi)) \setminus \{x_0\}) \in \wp_{\omega}(V_{\omega})$. Let $n \triangleq |V| \in \omega$ and $\bar{v}$ any enumeration of V. Likewise, $U \triangleq (\bigcup \mathrm{Var}[\{\phi, \psi\}]) \in \wp_{\omega}(V_{\omega})$. Take any $\bar{u} \in (V_{\omega} \backslash U)^n$. Then, by the reflexivity of θ, we have $\xi \triangleq (\sigma_{1:+1}(\varphi)[x_0/\phi; v_i/u_i]_{i \in n})\ \theta\ \eta \triangleq (\sigma_{1:+1}(\varphi)[x_0/\psi; v_i/u_i]_{i \in n})$. Let $f \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ extend $(h \upharpoonright U) \cup [u_i/g(v_i)]_{i \in n}$. Then, as $\mathcal{A} \in \mathrm{Mod}(C)$ and $\theta \subseteq \equiv^{\omega}_{C}$, we get $g(\sigma_{1:+1}(\varphi)) = f(\xi)\ \theta^{\mathcal{A}}\ f(\eta) = g(\sigma_{1:+1}(\varphi)[x_0/x_1])$. In this way, $h(\phi)\ \theta^{\mathcal{A}}_{\varepsilon_{\mathrm{Fm}^{\omega}_{\Sigma}}}\ h(\psi)$, and so Lemma 3.2 completes the argument. □

As a particular case of Lemma 3.7, we have:

Corollary 3.8. *Let C be a self-extensional Σ-logic and $\mathcal{A} \in \mathrm{Mod}^*(C)$. Then, $\mathfrak{A} \in \mathrm{IV}(C)$.*

Theorem 3.9. *Let M be a class of simple Σ-matrices, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$, $\mathsf{V} \triangleq \mathbf{V}(\mathsf{K})$, $\alpha \triangleq (1 \cup (\omega \cap \bigcup\{|A| \mid \mathcal{A} \in \mathsf{M}\})) \in \wp_{\infty \backslash 1}(\omega)$ and C the logic of M. Then, the following are equivalent:*

(i) *C is self-extensional;*
(ii) *for all $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma}$, it holds that $(\phi \equiv^{\omega}_{C} \psi) \Rightarrow (\mathsf{K} \models (\phi \approx \psi))$;*
(iii) *for all $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma}$, it holds that $(\phi \equiv^{\omega}_{C} \psi) \Leftrightarrow (\mathsf{K} \models (\phi \approx \psi))$;*
(iv) *for all distinct $a, b \in F^{\alpha}_{\Sigma}$, there are some $\mathcal{A} \in \mathsf{M}$ and some $h \in \hom(\mathfrak{Fm}^{\alpha}_{\mathsf{V}}, \mathfrak{A})$ such that $\chi^{\mathcal{A}}(h(a)) \neq \chi^{\mathcal{A}}(h(b))$;*
(v) *there is some class C of Σ-algebras such that $\mathsf{K} \subseteq \mathbf{V}(\mathsf{C})$ and, for each $\mathfrak{A} \in \mathsf{C}$ and all distinct $a, b \in A$, there are some $\mathcal{B} \in \mathsf{M}$ and some $h \in \hom(\mathfrak{A}, \mathfrak{B})$ such that $\chi^{\mathcal{B}}(h(a)) \neq \chi^{\mathcal{B}}(h(b))$;*
(vi) *there is some $\mathsf{S} \subseteq \mathrm{Mod}(C)$ such that $\mathsf{K} \subseteq \mathbf{V}(\pi_0[\mathsf{S}])$ and, for each $\mathcal{A} \in \mathsf{S}$, it holds that $(A^2 \cap \bigcap\{\theta^{\mathcal{B}} \mid \mathcal{B} \in \mathsf{S}, \mathfrak{B} = \mathfrak{A}\}) \subseteq \Delta_A$;*

in which case $\mathrm{IV}(C) = \mathsf{V}$.

Proof. First, (i)⇒(ii) is by Lemma 3.7.

Next, assume (ii) holds. Consider any $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma}$ such that $\mathsf{K} \models (\phi \approx \psi)$. Then, for each $\mathcal{A} \in \mathsf{M}$ and every $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$, $\langle h(\psi), h(\phi) \rangle \in \Delta_A \subseteq \theta^{\mathcal{A}}$, in which case $\psi \equiv^{\omega}_{C} \psi$, and so (iii) holds.

Further, assume (iii) holds. Then, $\theta^{\beta} \triangleq \theta^{\beta}_{C} = \theta^{\beta}_{\mathsf{K}} = \theta^{\beta}_{\mathsf{V}} \in \mathrm{Con}(\mathfrak{Fm}^{\beta}_{\Sigma})$, for all $\beta \in \wp_{\infty \backslash 1}(\omega)$. In particular (when $\beta = \omega$), we conclude that (i) holds, while $\mathrm{IV}(C) = \mathsf{V}$. Furthermore, consider any distinct $a, b \in F^{\alpha}_{\mathsf{V}}$. Then, there are some $\phi, \psi \in \mathrm{Fm}^{\alpha}_{\Sigma}$ such that $\nu_{\theta^{\alpha}}(\phi) = a \neq b = \nu_{\theta^{\alpha}}(\phi)$, in which case, by (2.12), $\mathrm{Cn}^{\alpha}_{\mathsf{M}}(\phi) \neq \mathrm{Cn}^{\alpha}_{\mathsf{M}}(\psi)$, and so there are some $\mathcal{A} \in \mathsf{M}$ and some $g \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})$ such that $\chi^{\mathcal{A}}(g(\phi)) \neq \chi^{\mathcal{A}}(g(\phi))$. In that case, $\theta^{\alpha} \subseteq (\ker g)$, and so, by the Homomorphism Theorem, $h \triangleq (g \circ \nu^{-1}_{\theta^{\alpha}}) \in \hom(\mathfrak{F}^{\alpha}_{\mathsf{V}}, \mathfrak{A})$. Then, $h(a/b) = g(\phi/\psi)$, in which case $\chi^{\mathcal{A}}(h(a)) \neq \chi^{\mathcal{A}}(h(b))$, and so (iv) holds.

Now, assume (iv) holds. Let $\mathsf{C} \triangleq \{\mathfrak{F}^{\alpha}_{\mathsf{V}}\}$. Consider any $\mathfrak{A} \in \mathsf{K}$ and the following complementary cases:

- $|A| \leqslant \alpha$.
  Let $h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})$ extend any surjection from V_{α} onto A, in which case

it is surjective, while $\theta \triangleq \theta^{\alpha}_{\mathsf{V}} = \theta^{\alpha}_{\mathsf{K}} \subseteq (\ker h)$, and so, by the Homomorphism Theorem, $g \triangleq (h \circ \nu_{\theta}^{-1}) \in \hom(\mathfrak{F}^{\alpha}_{\mathsf{V}}, \mathfrak{A})$ is surjective. In this way, $\mathfrak{A} \in \mathbf{V}(\mathfrak{F}^{\alpha}_{\mathsf{V}})$.
- $|A| \nleqslant \alpha$.
Then, $\alpha = \omega$. Consider any Σ-identity $\phi \approx \psi$ true in $\mathfrak{F}^{\omega}_{\mathsf{V}}$ and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$, in which case, we have $\theta \triangleq \theta^{\omega}_{\mathsf{V}} = \theta^{\omega}_{\mathsf{K}} \subseteq (\ker h)$, and so, since $\nu_{\theta} \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{F}^{\omega}_{\mathsf{V}})$, we get $\langle \phi, \psi \rangle \in (\ker \nu_{\theta}) \subseteq (\ker h)$. In this way, $\mathfrak{A} \in \mathbf{V}(\mathfrak{F}^{\alpha}_{\mathsf{V}})$.

Thus, $\mathsf{K} \subseteq \mathbf{V}(\mathsf{C})$, and so (v) holds.

Then, assume (v) holds. Let C' be the class of all non-one-element members of C and $\mathsf{S} \triangleq \{\langle \mathfrak{A}, h^{-1}[D^{\mathcal{B}}]\rangle \mid \mathfrak{A} \in \mathsf{C}', \mathcal{B} \in \mathsf{M}, h \in \hom(\mathfrak{A}, \mathfrak{B})\}$. Then, for all $\mathfrak{A} \in \mathsf{C}'$, each $\mathcal{B} \in \mathsf{M}$ and every $h \in \hom(\mathfrak{A}, \mathfrak{B})$, h is a strict homomorphism from $\mathcal{C} \triangleq \langle \mathfrak{A}, h^{-1}[D^{\mathcal{B}}]\rangle$ to $\mathcal{B}$, in which case, by (2.13), $\mathcal{C} \in \mathrm{Mod}(C)$, and so $\mathsf{S} \subseteq \mathrm{Mod}(C)$, while $\chi^{\mathcal{C}} = (\chi^{\mathcal{B}} \circ h)$, whereas $\pi_0[\mathsf{S}] = \mathsf{C}'$ generates the variety $\mathbf{V}(\mathsf{C})$. In this way, (vi) holds.

Finally, assume (vi) holds. Consider any $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma}$ such that $\phi \equiv^{\omega}_{C} \psi$. Consider any $\mathcal{A} \in \mathsf{S}$ and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$. Then, for each $\mathcal{B} \in \mathsf{S}$ with $\mathfrak{B} = \mathfrak{A}$, $h(\phi)\ \theta^{\mathcal{B}}\ h(\psi)$, in which case $h(\phi) = h(\psi)$, and so $\mathfrak{A} \models (\phi \approx \psi)$. Thus, (ii) holds. □

When both M and all members of it are finite, α is finite, in which case $\mathfrak{F}^{\alpha}_{\mathsf{V}}$ is finite and can be found effectively, and so the item (iv) of Theorem 3.9 yields an effective procedure of checking the self-extensionality of C. However, it computational complexity may be too large to count it *practically* applicable. For instance, in the n-valued case, where $n \in \omega$, the upper limit n^{n^n} of $|F^{\alpha}_{\mathsf{V}}|$ predetermining the computational complexity of the procedure involved becomes too large even in the tree-/four-valued case. And, though in the two-valued case this limit — 16 — is reasonably acceptable, this is no longer matter in view of the following generic observation:

Example 3.10. Let $\mathcal{A}$ be a Σ-matrix. Suppose it is both false- and truth-singular (in particular, two-valued as well as both consistent and truth-non-empty [in particular, classical]), in which case $\theta^{\mathcal{A}} = \Delta_A$, for $\chi^{\mathcal{A}}$ is injective, and so $\mathcal{A}$ is simple. Then, by Theorem 3.9(vi)⇒(i) with $\mathsf{S} = \{\mathcal{A}\}$, the logic of $\mathcal{A}$ is self-extensional, its intrinsic variety being generated by $\mathfrak{A}$. In this way, by the self-extensionality of inferentially inconsistent logics, any two-valued (in particular, classical) logic is self-extensional. □

Nevertheless, the procedure involved is simplified much under certain conditions upon the basis of the item (v) of Theorem 3.9.

3.3.1. *Self-extensional conjunctive disjunctive logics.* A Σ-algebra $\mathfrak{B}$ is called a $\bar{\wedge}$-*semi-lattice*, provided it satisfies semilattice identities for $\bar{\wedge}$, in which case we have the partial ordering $\leq^{\mathfrak{B}}_{\bar{\wedge}}$ on B, given by $(a \leq^{\mathfrak{B}}_{\bar{\wedge}} b) \stackrel{\text{def}}{\Longleftrightarrow} (a = (a \mathbin{\bar{\wedge}^{\mathfrak{A}}} b))$, for all $a, b \in B$. Then, in case B is finite, the poset $\langle B, \leq^{\mathfrak{B}}_{\bar{\wedge}}\rangle$ has the least element (zero) $\flat^{\mathfrak{B}}_{\bar{\wedge}}$. Likewise, $\mathfrak{B}$ is called a *[distributive]* $(\bar{\wedge}, \veebar)$-*lattice*, provided it satisfies [distributive] lattice identities for $\bar{\wedge}$ and $\veebar$, in which case $\leq^{\mathfrak{B}}_{\bar{\wedge}}$ and $\leq^{\mathfrak{B}}_{\veebar}$ are inverse to one another, and so, in case B is finite, $\flat^{\mathfrak{B}}_{\veebar}$ is the greatest element (unit) of the poset $\langle B, \leq^{\mathfrak{B}}_{\bar{\wedge}}\rangle$.

Lemma 3.11. *Let C' be a [finitary $\bar{\wedge}$-conjunctive] Σ-logic and $\mathcal{B}$ a [truth-non-empty $\bar{\wedge}$-conjunctive] Σ-matrix. Then, $\mathcal{B} \in \mathrm{Mod}_{2\setminus 1}(C')$ if[f] $\mathcal{B} \in \mathrm{Mod}(C')$.*

Proof. The "if" part is trivial. [Conversely, assume $\mathcal{B} \in \mathrm{Mod}_{2\setminus 1}(C)$. Then, by Remark 2.3, $\mathcal{B} \in \mathrm{Mod}_2(C')$. By induction on any $n \in \omega$, let us prove that $\mathcal{B} \in \mathrm{Mod}_n(C')$. For consider any $X \in \wp_n(\mathrm{Fm}^{\omega}_{\Sigma})$, in which case $n \neq 0$. The case, when $|X| \in 2$, has been proved above. Now, assume $|X| \geqslant 2$, in which case there are some distinct $\phi,\psi \in X$, and so $Y \triangleq ((X \setminus \{\phi,\psi\}) \cup \{\phi \mathbin{\bar\wedge} \psi\}) \in \wp_{n-1}(\mathrm{Fm}^{\omega}_{\Sigma})$. Then, by the induction hypothesis and the $\bar\wedge$-conjunctivity of both C' and $\mathcal{B}$, we get $C'(X) = C'(Y) \subseteq \mathrm{Cn}^{\omega}_{\mathcal{B}}(Y) = \mathrm{Cn}^{\omega}_{\mathcal{B}}(X)$. Thus, $\mathcal{B} \in \mathrm{Mod}_{\omega}(C')$, for $\omega = (\bigcup\omega)$, and so $\mathcal{B} \in \mathrm{Mod}(C')$, for C' is finitary.] □

Corollary 3.12. *Let M be a class of simple Σ-matrices, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$, $\mathsf{V} \triangleq \mathbf{V}(\mathsf{K})$ and C the logic of M. Suppose C is finitary (in particular, both M and all members of it are finite) and $\bar\wedge$-conjunctive (that is, all members of M are so) [as well as $\veebar$-disjunctive (in particular, all members of M are so)]. Then, the following are equivalent:*

(i) *C is self-extensional;*
(ii) *for all $\phi,\psi \in \mathrm{Fm}^{\omega}_{\Sigma}$, it holds that $(\psi \in C(\phi)) \Leftrightarrow (\mathsf{K} \models (\phi \approx (\phi \mathbin{\bar\wedge} \psi)))$, while semilattice [more generally, distributive lattice] identities for $\bar\wedge$ [and $\veebar$] are true in K;*
(iii) *every truth-non-empty $\bar\wedge$-conjunctive [consistent $\veebar$- disjunctive] Σ-matrix with underlying algebra in V is a model of C, while semilattice [more generally, distributive lattice] identities for $\bar\wedge$ [and $\veebar$] are true in K;*
(iv) *every truth-non-empty $\bar\wedge$-conjunctive [consistent $\veebar$- disjunctive] Σ-matrix with underlying algebra in K is a model of C, while semilattice [more generally, distributive lattice] identities for $\bar\wedge$ [and $\veebar$] are true in K.*

Proof. First, it is routine checking that, for every semilattice [more generally, distributive lattice] identity $\phi \approx \psi$ for $\bar\wedge$ [and $\veebar$], it holds that $\phi \approx^{\omega}_{C} \psi$. In this way, (i)⇒(ii) is by Theorem 3.9(i)⇒(iii) and the $\bar\wedge$-conjuctivity of C. Next, (ii)⇒(iii) is by Lemma 3.11. Further, (iv) is a particular case of (iii). Finally, (iv)⇒(i) is by Theorem 3.9(vi)⇒(i) with S, being the class of all truth-non-empty $\bar\wedge$-conjunctive [consistent $\veebar$- disjunctive] Σ-matrices with underlying algebra in K, and the semilattice identities for $\bar\wedge$ [as well as the Prime Ideal Theorem for distributive lattices]. (More precisely, consider any $\mathfrak{A} \in \mathsf{K}$ and any $\bar{a} \in (A^2 \setminus \Delta_A)$, in which case, by the semilattice identities for $\bar\wedge$, $a_i \neq (a_i \mathbin{\bar\wedge^{\mathfrak{A}}} a_{1-i})$, for some $i \in 2$, and so $\mathcal{B} \triangleq \langle \mathfrak{A}, \{b \in A \mid a_i = (a_i \mathbin{\bar\wedge^{\mathfrak{A}}} b)\}\rangle \in \mathsf{S}$ [resp., by the Prime Ideal Theorem, there is some $\mathcal{B} \in \mathsf{S}$] such that $a_i \in D^{\mathcal{B}} \not\ni a_{1-i}$.) □

Corollary 3.13. *Let M be a finite class of finite hereditarily simple $\bar\wedge$-conjunctive $\veebar$-disjunctive Σ-matrices, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ and C the logic of M. Then, C is self-extensional iff, for each $\mathfrak{A} \in \mathsf{K}$ and all distinct $a,b \in A$, there are some $\mathcal{B} \in \mathsf{M}$ and some non-singular $h \in \hom(\mathfrak{A},\mathfrak{B})$ such that $\chi^{\mathcal{B}}(h(a)) \neq \chi^{\mathcal{B}}(h(b))$.*

Proof. The "if" part is by Theorem 3.9(v)⇒(i) with $\mathsf{C} = \mathsf{K}$. Conversely, assume C is self-extensional. Consider any $\mathfrak{A} \in \mathsf{K}$ and any $\bar{a} \in (A^2 \setminus \Delta_A)$. Then, by Corollary 3.12(i)⇒(iv), $\mathfrak{A}$ is a distributive $(\bar\wedge,\veebar)$-lattice, in which case, by the commutativity identity for $\bar\wedge$, $a_i \neq (a_i \mathbin{\bar\wedge^{\mathfrak{A}}} a_{1-i})$, for some $i \in 2$, and so, by the Prime Ideal Theorem, there is some $\bar\wedge$-conjunctive $\veebar$-disjunctive Σ-matrix $\mathcal{D}$ with $\mathfrak{D} = \mathfrak{A}$ such that $a_i \in D^{\mathcal{D}} \not\ni a_{1-i}$, in which case $\mathcal{D}$ is both consistent and truth-non-empty, and so is a model of C. Hence, by Lemmas 2.7, 2.8 and Remark 2.5, there are some $\mathcal{B} \in \mathsf{M}$ and some $h \in \hom_{\mathsf{S}}(\mathcal{D},\mathcal{B}) \subseteq \hom(\mathfrak{A},\mathfrak{B})$, in which case $h(a_i) \in D^{\mathcal{B}} \not\ni h(a_{1-i})$, and so $\chi^{\mathcal{B}}(h(a_i)) = 1 \neq 0 = \chi^{\mathcal{B}}(h(a_{1-i}))$, while, as $h(a_i) \neq h(a_{1-i})$, h is not singular, as required. □

The effective procedure of verifying the self-extensionality of an n-valued disjunctive conjunctive logic, where $n \in \omega$, resulted from Corollary 3.13 has the computational complexity n^n that is quite acceptable for three-/four-valued logics. And what is more, it provides a quite useful heuristic tool of doing it, manual applications of which are presented below. First, we have:

Corollary 3.14. *Let $n \in (\omega \setminus 3)$, $\mathcal{A}$ a hereditarily simple $\overline{\wedge}$-conjunctive $\veebar$-disjunctive Σ-matrix and C the logic of $\mathcal{A}$. Suppose every non-singular endomorphism of $\mathfrak{A}$ is diagonal. Then, the logic of $\mathcal{A}$ is not self-extensional.*

Proof. By contradiction. For suppose C is self-extensional. Then, $\mathcal{A}$ is either false- or truth-non-singular, in which case $\chi^{\mathcal{A}}$ is not injective, and so there are some distinct $a, b \in A$ such that $\chi^{\mathcal{A}}(a) = \chi^{\mathcal{A}}(b)$. On the other hand, by Corollary 3.13, there is some non-singular $h \in \hom(\mathfrak{A}, \mathfrak{A})$ such that $\chi^{\mathcal{A}}(h(a)) \neq \chi^{\mathcal{A}}(h(b))$, in which case $h = \Delta_A$, and so $\chi^{\mathcal{A}}(a) = \chi^{\mathcal{A}}(h(a)) \neq \chi^{\mathcal{A}}(h(b)) = \chi^{\mathcal{A}}(b)$. This contradiction completes the argument. □

As a consequence, by Theorem 14 of [16] and Corollaries 3.5 and 3.14, we immediately get the following universal negative result:

Corollary 3.15. *Let $n \in (\omega \setminus 3)$, $\mathcal{A}$ a $\overline{\wedge}$-conjunctive $\veebar$-disjunctive Σ-matrix with unary unitary equality determinant, C the logic of $\mathcal{A}$ and $\widetilde{\mathcal{S}}^{(k,l)}_{\mathcal{A},\mathcal{T}}$ as in Theorem 14 of* [16]. *Suppose $\widetilde{\mathcal{S}}^{(k,l)}_{\mathcal{A},\mathcal{T}}$ is algebraizable. Then, C is not self-extensional.*

In particular, we have:

Example 3.16 (Finitely-valued Łukasiewicz' logics; cf. [6])**.** Let $n \in (\omega \setminus 2)$, $\Sigma \triangleq (\Sigma_+ \cup \{\supset, \sim\})$ and $\mathcal{A}$ the Σ-matrix with $A \triangleq (n \div (n-1))$, $D^{\mathcal{A}} \triangleq \{1\}$, $\sim^{\mathfrak{A}} \triangleq (1-a)$, $(a \wedge^{\mathfrak{A}} b) \triangleq \min(a, b)$, $(a \vee^{\mathfrak{A}} b) \triangleq \max(a, b)$ and $(a \supset^{\mathfrak{A}} b) \triangleq \min(1, 1-a+b)$, for all $a, b \in A$, in which case $\mathcal{A}$ is both $\wedge$-conjunctive and $\veebar$-disjunctive as well as has a unary unitary equality determinant, by Example 3 of [15]. And what is more, by Example 7 of [16], $\widetilde{\mathcal{S}}^{(k,l)}_{\mathcal{A},\mathcal{T}}$ is algebraizable. Hence, by Corollary 3.15, the logic of $\mathcal{A}$ is not self-extensional. □

A one more universal application is discussed below.
3.3.1.1. Application to four-valued expansions of the least De Morgan logic. Here, it is supposed that $\Sigma \supseteq \Sigma_{\sim,+[,01]} \triangleq (\Sigma_{+[,01]} \cup \{\sim\})$. Fix a Σ-matrix $\mathcal{A}$ with $A \triangleq 2^2$, $D^{\mathcal{A}} \triangleq (2^2 \cap \pi_0^{-1}[\{1\}]$, $\mathfrak{A}{\restriction}\Sigma_{+[,01]}) \triangleq \mathfrak{D}^2_{2[,01]}$ and $\sim^{\mathfrak{A}}\langle i, j\rangle \triangleq \langle 1-j, 1-i\rangle$, for all $i, j \in 2$. Then, $\mathcal{A}$ is both $\wedge$-conjunctive and $\veebar$-disjunctive, while $\{x_0, \sim x_0\}$ is a unary unitary equality determinant for it (cf. Example 2 of [15]), so it is hereditarily simple (cf. Corollary 3.5). Let C be the logic of $\mathcal{A}$. Then, as $\mathcal{DM}_{4[,01]} \triangleq (\mathcal{A}{\restriction}\Sigma_{\sim,+[,01]})$ defines [the bounded version/expansion of] the least De Morgan logic $D_{4[,01]}$ (cf. [10] and the reference [Pyn 95a] of [11]), C is a four-valued expansion of $D_{4[,01]}$.

Let $\mu : 2^2 \to 2^2, \langle i, j\rangle \mapsto \langle j, i\rangle$ and $\sqsubseteq \triangleq \{\langle ij, kl\rangle \in (2^2)^2 \mid i \leqslant k, l \leqslant j\}$, commuting with μ/monotonic with respect to $\sqsubseteq$ operations on 2^2 being said to be *specular/regular*, respectively. Then, $\mathfrak{A}$ is said to be *specular/regular*, whenever its primary operations are so, in which case secondary ones are so as well. (Clearly, $\mathfrak{DM}_{4[,01]}$ is both specular and regular.)

Theorem 3.17. *C is self-extensional iff $\mathfrak{A}$ is specular.*

Proof. Note that, for all $a, b \in A$, it holds that $a = b$ iff both $(\pi_0(a) = \pi_0(b)) \Leftrightarrow ((a \in D^{\mathcal{A}}) \Leftrightarrow (b \in D^{\mathcal{A}}))$ and $\pi_0(\mu(a)) = \pi_1(a) = \pi_1(b) = \pi_0(\mu(b))$. In this way, the "if" part is by that of Corollary 3.13. Conversely, assume C is self-extensional. Then, by Corollary 3.13, there is some non-singular $h \in \hom(\mathfrak{A}, \mathfrak{A})$ such that $\chi^{\mathcal{A}}(h(11)) \neq \chi^{\mathcal{A}}(h(10))$, in which case $B \triangleq (\operatorname{img} h)$ forms a non-one-element subalgebra of $\mathfrak{A}$, and so $\Delta_2 \subseteq B$. Hence, $\langle 0/1, 0/1\rangle$ is zero/unit of $(\mathfrak{A}{\restriction}\Sigma_+)[{\restriction}B]$, in which case $(h{\restriction}\Delta_2)$ is diagonal, and so $h(10) \notin D^{\mathcal{A}}$. On the other hand, for all $a \in A$, it holds that $(\sim^{\mathfrak{A}} a = a) \Leftrightarrow (a \notin \Delta_2)$. Therefore, $h(10) = (01)$. Moreover, if $h(01)$ was equal to 01 too, then we would have $(00) = h(00) = h((10) \wedge^{\mathfrak{A}} (01)) = ((01) \wedge^{\mathfrak{A}} (01)) = (01)$. Thus, $\hom(\mathfrak{A}, \mathfrak{A}) \ni h = \mu$, as required. □

This positively covers $D_{4[,01]}$ as regular instances. And what is more, in case $\Sigma = \Sigma_{\simeq,+[,01]} \triangleq (\Sigma_{\sim,+[,01]} \cup \{\neg\})$ with unary $\neg$ (classical — viz., Boolean — negation) and $\neg^{\mathfrak{A}}\langle i,j\rangle \triangleq \langle 1-i, 1-j\rangle$, it equally covers the logic $CD_{4[,01]} \triangleq C$ of the $(\neg x_0 \vee x_1)$-implicative $\mathcal{DMB}_{4[,01]} \triangleq \mathcal{A}$ with non-regular underlying algebra, introduced in [13]. Below, we disclose a *unique* (up to term-wise definitional equivalence) status of these three instances.

Lemma 3.18. *Suppose $\mathfrak{A}$ is specular. Then, Δ_A forms a subalgebra of $\mathfrak{A}$.*

Proof. By contradiction. For suppose there are some $f \in \Sigma$ of arity $n \in \omega$ and some $\bar{a} \in \Delta_2^n$ such that $f^{\mathfrak{A}}(\bar{a}) \notin \Delta_2$. Then, $f^{\mathfrak{A}}(\bar{a}) = f^{\mathfrak{A}}(\mu \circ \bar{a}) = \mu(f^{\mathfrak{A}}(\bar{a})) \neq f^{\mathfrak{A}}(\bar{a})$. This contradiction completes the argument. □

Lemma 3.19. *Let C' be a Σ-logic, $\mathcal{B} \in \mathrm{Mod}^*(C')$ and $\phi, \psi \in C'(\varnothing)$. Suppose C' is self-extensional. Then, $\mathfrak{B} \models (\phi \approx \psi)$.*

Proof. In that case, $\phi \equiv^{\omega}_{C'} \psi$, and so Corollary 3.8 completes the argument. □

Corollary 3.20. *Suppose C is self-extensional. Then, the following are equivalent:*

(i) *C has a theorem;*
(ii) *$\top$ is term-wise definable in $\mathfrak{A}$;*
(iii) *$\bot$ is term-wise definable in $\mathfrak{A}$;*
(iv) *$\{01\}$ does not form a subalgebra of $\mathfrak{A}$;*
(v) *$\{10\}$ does not form a subalgebra of $\mathfrak{A}$.*

Proof. First, (i,iv) are particular cases of (ii), for $(01) \neq \top = (11) \in D^{\mathcal{A}}$. Next, (ii)⇔(iii) is by the equalities $\sim(\bot/\top) = (\top/\bot)$. Likewise, (iv)⇔(v) is by the equalities $\mu[\{01/10\}] = \{10/01\}$. Further, (i)⇒(ii) is by Lemmas 3.18 and 3.19. Finally, assume (iv) holds. Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(01) \neq (01)$, in which case, by Theorem 3.17 and the injectivity of μ, we have $(10) = \mu(01) \neq \mu(\varphi^{\mathfrak{A}}(01)) = \varphi^{\mathfrak{A}}(\mu(01)) = \varphi^{\mathfrak{A}}(10)$, and so, by Lemma 3.18, we get $(x_0 \vee (\varphi \vee \sim\varphi)) \in C(\varnothing)$. Thus, (i) holds, as required. □

Corollary 3.21. *Suppose C is self-extensional, and $\mathcal{A}$ is $\sqsupset$-implicative. Then, $\neg$ is term-wise definable in $\mathfrak{A}$.*

Proof. Then, by (2.6), true in $\mathcal{A}$, and Corollary 3.20, $\bot$ is term-wise definable in $\mathfrak{A}$ (more precisely, as $\sim(x_0 \sqsupset x_0)$), and so $\mathcal{A}$ is $-$-negative, where $-x_0 \triangleq (x_0 \sqsupset \bot)$. Moreover, by Theorem 3.17, $\mathfrak{A}$ is specular, in which case, by Lemma 3.18, Δ_2 forms a subalgebra of $\mathfrak{A}$, and so $(-^{\mathfrak{A}}{\restriction}\Delta_2) = (\neg{\restriction}\Delta_2)$. On the other hand, if $-^{\mathfrak{A}}(10) \notin D^{\mathcal{A}}$ was equal to 00, then we would have $D^{\mathcal{A}} \ni -^{\mathfrak{A}}(01) = -^{\mathfrak{A}}(\mu(01)) = \mu(-^{\mathfrak{A}}(10)) = \mu(00) = (00) \notin D^{\mathcal{A}}$. Therefore, $-^{\mathfrak{A}}(10) = (01)$, in which case $(10) = \mu(01) = \mu(-^{\mathfrak{A}}(10)) = -^{\mathfrak{A}}\mu(10) = -^{\mathfrak{A}}(01)$, and so $-^{\mathfrak{A}} = \neg$, as required. □

3.3.1.1.1. Specular functional completeness. As usual, *Boolean algebras* are supposed to be of the signature $\Sigma^- \triangleq (\Sigma_{\simeq,+,01} \setminus \{\sim\})$, the ordinary one over 2 being denoted by $\mathfrak{B}_2$.

Lemma 3.22. *Let $n \in \omega$ and $f : 2^n \to 2$. [Suppose f is monotonic with respect to $\leqslant$ (and $f(n \times \{i\}) = i$, for each $i \in 2$, in which case $n > 0$).] Then, there is some $\vartheta \in \mathrm{Fm}^n_{\Sigma^-[\setminus\{\neg\}(\setminus\{\bot,\top\})]}$ such that $g = \vartheta^{\mathfrak{B}_2}$.*

Proof. Then, by the functional completeness of $\mathfrak{B}_2$, there is some $\vartheta \in \mathrm{Fm}^n_{\Sigma^-}$ such that $g = \vartheta^{\mathfrak{B}_2}(\notin \{2^n \times \{i\} \mid i \in 2\})$, in which case, without loss of generality, one can assume that $\vartheta = (\wedge\langle\vec{\varphi}, \top\rangle)$, where, for each $m \in \ell \triangleq (\mathrm{dom}\,\vec{\varphi}) \in (\omega(\setminus 1))$, $\varphi_m = (\vee\langle(\neg \circ \vec{\phi}^m) * \vec{\psi}^m, \bot\rangle)$, for some $\vec{\phi}^m \in V_n^{k_m}$, some $\vec{\psi}^m \in V_n^{l_m}$ and some $k_m, l_m \in \omega$ such that $((\mathrm{img}\,\vec{\phi}^m) \cap (\mathrm{img}\,\vec{\psi}^m)) = \varnothing$. [Set $\vartheta'' \triangleq (\wedge\langle\vec{\varphi}'', \top\rangle)$, where, for each $m \in (\operatorname{dom}\vec{\varphi}'') \triangleq \ell$, $\varphi''_m \triangleq (\vee\langle\vec{\psi}^m, \bot\rangle)$. Consider any $\bar{a} \in A^n$ and the following exhaustive cases:

(1) $g(\bar{a}) = 0$,
in which case we have $\vartheta''^{\mathfrak{B}_2}[x_j/a_j]_{j\in n} \leqslant \vartheta^{\mathfrak{B}_2}[x_j/a_j]_{j\in n} = 0$, and so we get $\vartheta''^{\mathfrak{B}_2}[x_j/a_j]_{j\in n} = 0$.

(2) $g(\bar{a}) = 1$,
in which case, for every $m \in \ell$, as $\bar{a} \leqslant \bar{b} \triangleq ((\bar{a}{\restriction}(n \setminus N)) \cup (N \times \{1\})) \in A^n$, where $N \triangleq \{j \in n \mid x_j \in (\operatorname{img}\vec{\phi}^m)\}$, by the $\leqslant$-monotonicity of g, we have $1 \leqslant g(\bar{b}) \leqslant \varphi^{\mathfrak{B}_2}_m[x_j/b_j]_{j\in n} = \varphi''^{\mathfrak{B}_2}_m[x_j/a_j]_{j\in n}$, and so we get $\vartheta''^{\mathfrak{B}_2}[x_j/a_j]_{j\in n} = 1$.

Thus, $g = \vartheta''^{\mathfrak{B}_2}$. (And what is more, since, in that case, $\ell > 0$ and $l_m > 0$, for each $m \in \ell$, we also have $g = \vartheta'''^{\mathfrak{B}_2}$, where $\vartheta''' \triangleq (\wedge\vec{\varphi}''')$, whereas, for each $m \in (\operatorname{dom}\vec{\varphi}''') \triangleq \ell$, $\varphi'''_m \triangleq (\vee\vec{\psi}^m)$.)] This completes the argument. □

Theorem 3.23. *Let $\Sigma = \Sigma_{\simeq,+,01}$, $n \in (\omega(\setminus 1))$ and $f : A^n \to A$. Then, f is specular [and regular (as well as $f(n \times \{a\}) = a$, for all $a \in (A \setminus \Delta_A)$)] iff there is some $\tau \in \mathrm{Fm}^n_{\Sigma[\setminus\{\neg\}(\setminus\{\bot,\top\})]}$ such that $f = \tau^{\mathfrak{A}}$.*

Proof. The "if" part is immediate. Conversely, assume f is specular [and regular (as well as $f(n \times \{a\}) = a$, for all $a \in (A \setminus \Delta_A)$)]. Then,

$$g : 2^{2\cdot n} \to 2, \bar{a} \mapsto \pi_0(f(\langle\langle a_{2\cdot j}, 1 - a_{(2\cdot j)+1}\rangle\rangle_{j\in n}))$$

[is monotonic with resect to $\leqslant$ (and $g(n \times \{i\}) = i$, for each $i \in 2$)]. Therefore, by Lemma 3.22, there is some $\vartheta \in \mathrm{Fm}^{2\cdot n}_{\Sigma^-[\setminus\{\neg\}(\setminus\{\bot,\top\})]}$ such that $g = \vartheta^{\mathfrak{B}_2}$. Put

$$\tau \triangleq (\vartheta[x_{2\cdot j}/x_j, x_{(2\cdot j)+1}/(\sim x_j)]_{j\in n}) \in \mathrm{Fm}^n_{\Sigma[\setminus\{\neg\}](\setminus\{\bot,\top\})}.$$

Consider any $\bar{c} \in A^n$. Then, since, for each $i \in 2$, we have $\pi_i \in \hom(\mathfrak{A}{\restriction}\Sigma^-, \mathfrak{B}_2)$, we get $\pi_0(\tau^{\mathfrak{A}}[x_j/c_j]_{j\in n}) = \vartheta^{\mathfrak{B}_2}[x_{2\cdot j}/\pi_0(c_j), x_{(2\cdot j)+1}/(1 - \pi_1(c_j))]_{j\in n} = \pi_0(f(\bar{c}))$ and, likewise, as f is specular, $\pi_1(\tau^{\mathfrak{A}}[x_j/c_j]_{j\in n}) = \vartheta^{\mathfrak{B}_2}[x_{2\cdot j}/\pi_1(c_j), x_{(2\cdot j)+1}/(1 - \pi_0(c_j))]_{j\in n} = \pi_0(f(\mu \circ \bar{c})) = \pi_0(\mu(f(\bar{c}))) = \pi_1(f(\bar{c}))$. □

In this way, by Theorems 3.17 and 3.23, $CD_{4[,01]}$ is the most expansive (up to term-wise definitional equivalence) self-extensional four-valued expansion of D_4. And what is more, combining Theorems 3.17 and 3.23 with Corollaries 3.20 and 3.21, we eventually get:

Corollary 3.24. *C is self-extensional, while $\mathcal{A}$ is implicative/both $\mathfrak{A}$ is regular and C is [not] purely-inferential, iff C is term-wise definitionally equivalent to $CD_4/D_{4[,01]}$, respectively.*

3.4. **Disjunctive extensions of disjunctive finitely-valued logics.** Given any $X, Y \subseteq \mathrm{Fm}^\omega_\Sigma$, put $(X \veebar Y) \triangleq \veebar[X \times Y]$.

Lemma 3.25. *Let C be a $\veebar$-disjunctive Σ-logic. Then,*

$$(\varphi \veebar C(X \cup Y)) \subseteq C(X \cup (\varphi \veebar Y)), \tag{3.2}$$

for all $X \subseteq \mathrm{Fm}^\omega_\Sigma$, all $\varphi \in \mathrm{Fm}^\omega_\Sigma$ and all $Y \in \wp_\omega(\mathrm{Fm}^\omega_\Sigma)$.

Proof. By induction on $|Y| \in \omega$. The case, when $Y = \varnothing$, is by (2.3) and (2.4). Now, assume $Y \neq \varnothing$. Take any $\psi \in Y$, in which case $X' \triangleq (X \cup \{\psi\}) \subseteq \mathrm{Fm}^\omega_\Sigma$ and $Y' \triangleq (Y \setminus \{\psi\}) \in \wp_\omega(\mathrm{Fm}^\omega_\Sigma)$, while $|Y'| < |Y|$, whereas $(Y' \cup X') = (X \cup Y)$, and so, by induction hypothesis, we have $(\varphi \veebar C(X \cup Y)) \subseteq C(X' \cup (\varphi \veebar Y'))$. On the other hand, by (2.3), we also have $(\varphi \veebar C(X \cup Y)) \subseteq C((X \cup \{\varphi\}) \cup (\varphi \veebar Y'))$. Thus, as $Y = (Y' \cup \{\psi\})$, the $\veebar$-disjunctivity of C yields (3.2). □

Given a Σ-rule $\Gamma \vdash \phi$ and a Σ-formula ψ, put $((\Gamma \vdash \phi) \veebar \psi) \triangleq ((\Gamma \veebar \psi) \vdash (\phi \veebar \psi))$. (This notation is naturally extended to Σ-calculi member-wise.)

Theorem 3.26. *Let M be a [finite] class of [finite $\veebar$-disjunctive] Σ-matrices, C the logic of M, while $\mathcal{A}$ an axiomatic Σ-calculus [whereas $\mathcal{C}$ a finitary Σ-calculus]. Then, the extension C' of C relatively axiomatized by $\mathcal{C}' \triangleq (\mathcal{A}[\cup(\sigma_{+1}[\mathcal{C}] \veebar x_0)])$ is defined by $\mathsf{S} \triangleq (\mathrm{Mod}(\mathcal{A}[\cup\mathcal{C}]) \cap \mathbf{S}_*(\mathsf{M}))$ [and so is $\veebar$-disjunctive].*

Proof. First, by (2.13) [and Lemma 3.25 with $X = \varnothing$ as well as the $\veebar$-disjunctivity of every $\mathcal{A} \in \mathbf{S}_*(\mathsf{M})$, and so both that and the structurality of $\mathrm{Cn}^{\omega}_{\mathcal{A}}$], we have $\mathsf{S} = (\mathrm{Mod}(\mathcal{A})[\cap \mathrm{Mod}(\mathcal{C})] \cap \mathbf{S}_*(\mathsf{M})) \subseteq (\mathrm{Mod}(\mathcal{C}') \cap \mathbf{S}_*(\mathsf{M})) \subseteq (\mathrm{Mod}(\mathcal{C}') \cap \mathrm{Mod}(C)) = \mathrm{Mod}(C')$.

Conversely, consider any [finitary] Σ-rule $\Gamma \vdash \varphi$ not satisfied in C', in which case $\varphi \notin T \triangleq C'(\Gamma) \in (\mathrm{img}\, C') \subseteq (\mathrm{img}\, \mathrm{Cn}^{\omega}_{\mathsf{M}})$, and so [by the finiteness of $(\Gamma \cup \{\varphi\}) \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$], there is some [finite] $\alpha \in \wp_{\omega \setminus 1}(\omega)$ such that $(\Gamma \cup \{\varphi\}) \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$, in which case $\Gamma \subseteq U \triangleq (T \cap \mathrm{Fm}^{\alpha}_{\Sigma}) \not\ni \varphi$, and so, by (2.12), $U = \mathrm{Cn}^{\alpha}_{\mathsf{M}}(U) = (\mathrm{Fm}^{\alpha}_{\Sigma} \cap \bigcap \mathcal{U})$, where $\mathcal{U} \triangleq \{h^{-1}[D^{\mathcal{A}}] \supseteq U \mid \mathcal{A} \in \mathsf{M}, h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})\}$ [is finite, for α as well as both M and all members of it are so]. Therefore, there is some [minimal] $S \in \mathcal{U}$ not containing φ, in which case, $\Gamma \subseteq U \subseteq S$, and so $\Gamma \vdash \varphi$ is not true in $\mathcal{B} \triangleq \langle \mathfrak{Fm}^{\alpha}_{\Sigma}, S\rangle$ under $[x_i/x_i]_{i \in \alpha}$. Next, we are going to show that $\mathcal{B} \in \mathrm{Mod}(\mathcal{A}[\cup\mathcal{C}])$. For consider any $(\Delta \vdash \phi) \in (\mathcal{A}[\cup\mathcal{C}])$ and any $\sigma \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{Fm}^{\alpha}_{\Sigma})$ such that $\sigma[\Delta] \subseteq S$ as well as the following exhaustive case[s]:

- $(\Delta \vdash \phi) \in \mathcal{A}$,
  in which case $\Delta = \varnothing$, and so, as $\phi \in \mathcal{A} \subseteq \mathcal{C}'$, by the structurality of C', we have $\sigma(\phi) \in (\mathrm{Fm}^{\alpha}_{\Sigma} \cap C'(\varnothing)) \subseteq (\mathrm{Fm}^{\alpha}_{\Sigma} \cap T) = U \subseteq S$.
- [$(\Delta \vdash \phi) \in \mathcal{C}$,
  in which case $((\sigma_{+1}[\Delta] \vdash \sigma_{+1}(\phi)) \veebar x_0) \in \mathcal{C}'$, and so is satisfied in C'. Then, $(\mathcal{U} \setminus \{S\}) \subseteq \mathcal{U}$ is finite, for $\mathcal{U}$ is so, in which case $n \triangleq |\mathcal{U} \setminus \{S\}| \in \omega$. Take any bijection $\overline{W} : n \to (\mathcal{U} \setminus \{S\})$. Then, for each $i \in n$, $W_n \neq S$, in which case, by the minimality of $S \in \mathcal{U} \ni W_n$, we have $W_n \nsubseteq S$, and so there is some $\xi_i \in (W_n \setminus S) \neq \varnothing$. Put $\psi \triangleq (\veebar\langle\bar{\xi}, \varphi\rangle) \in \mathrm{Fm}^{\alpha}_{\Sigma}$. Let ς be the Σ-substitution extending $[x_{i+1}/\sigma(x_i); x_0/\psi]_{i \in \omega}$. Then, $((\sigma[\Delta] \veebar \psi) \vdash (\sigma(\phi) \veebar \psi)) = \varsigma((\sigma_{+1}[\Delta] \vdash \sigma_{+1}(\phi)) \veebar x_0)$ is satisfied in C', for this is structural. Moreover, in view of the $\veebar$-disjunctivity of members of M, $(\sigma[\Delta] \veebar \psi) \subseteq (\mathrm{Fm}^{\alpha}_{\Sigma} \cap \bigcap \mathcal{U}) = U \subseteq T$, in which case $(\sigma(\phi) \veebar \psi) \in (\mathrm{Fm}^{\alpha}_{\Sigma} \cap T) = U \subseteq S$, and so $\sigma(\phi) \in S$, for $\psi \notin S$.]

Thus, $\mathcal{B} \in \mathrm{Mod}(\mathcal{A}[\cup\mathcal{C}])$. On the other hand, as $S \in \mathcal{U}$, there are some $\mathcal{A} \in \mathsf{M}$ and some $h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})$ such that $S = h^{-1}[D^{\mathcal{A}}]$, in which case $D \triangleq (\mathrm{img}\, h)$ forms a subalgebra of $\mathfrak{A}$, and so h is a surjective strict homomorphism from $\mathcal{B}$ onto $\mathcal{D} \triangleq (\mathcal{A} \upharpoonright D)$. In this way, by (2.13), $\Gamma \vdash \varphi$ is not true in $\mathcal{D} \in \mathsf{S}$, as required [for C' is finitary, as both C and $\mathcal{C}'$ are so]. □

Lemma 3.27. *Let C be a Σ-logic and M a finite class of finite Σ-matrices. Suppose C is finitely-defined by M. Then, C is defined by M, that is, C is finitary.*

Proof. In that case, $C' \triangleq \mathrm{Cn}^{\omega}_{\mathsf{M}} \subseteq C$, for C' is finitary. To prove the converse is to prove that $\mathsf{M} \subseteq \mathrm{Mod}(C)$. For consider any $\mathcal{A} \in \mathsf{M}$, any $\Gamma \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$, any $\varphi \in C(\Gamma)$ and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ such that $h[\Gamma] \subseteq D^{\mathcal{A}}$. Then, $\alpha \triangleq |A| \in (\wp_{\infty\setminus 1}(\omega) \cap \omega)$. Take any bijection $e : V_\alpha \to A$ to be extended to a $g \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})$. Then, $e^{-1} \circ (h \upharpoonright V_\omega)$ is extended to a Σ-substitution σ, in which case $\sigma(\varphi) \in C(\sigma[\Gamma])$, for C is structural, while $\sigma[\Gamma \cup \{\varphi\}] \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$. Further, as both α, M and all members of it are finite, we have the finite set $I \triangleq \{\langle f, \mathcal{B}\rangle \mid \mathcal{B} \in \mathsf{M}, f \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{B})\}$, in which case, for each $i \in I$, we set $h_i \triangleq \pi_0(i)$, $\mathcal{B}_i \triangleq \pi_1(i)$ and $\theta_i \triangleq \theta^{\mathcal{B}_i}$. Then, by (2.12), we have $\theta \triangleq \equiv^{\alpha}_{C} = \equiv^{\alpha}_{C'} = ((\mathrm{Fm}^{\alpha}_{\Sigma} \times \mathrm{Fm}^{\alpha}_{\Sigma}) \cap \bigcap_{i\in I} h_i^{-1}[\theta_i])$, in which case, for every $i \in I$, $\theta \subseteq h_i^{-1}[\theta_i] = \ker(\nu_{\theta_i} \circ h_i)$, and so $g_i \triangleq (\nu_{\theta_i} \circ h_i \circ \nu_{\theta}^{-1}) : (\mathrm{Fm}^{\alpha}_{\Sigma}/\theta) \to B_i$. In this way, $e \triangleq (\prod_{i\in I} g_i) : (\mathrm{Fm}^{\alpha}_{\Sigma}/\theta) \to (\prod_{i\in I} B_i)$ is injective, for $(\ker e) = ((\mathrm{Fm}^{\alpha}_{\Sigma}/\theta)^2 \cap \bigcap_{i\in I}(\ker g_i))$ is diagonal. Hence, $\mathrm{Fm}^{\alpha}_{\Sigma}/\theta$ is finite, for $\prod_{i\in I} B_i$ is so, and so is $(\sigma[\Gamma]/\theta) \subseteq (\mathrm{Fm}^{\alpha}_{\Sigma}/\theta)$. For each $c \in (\sigma[\Gamma]/\theta)$, choose any $\phi_c \in (\sigma[\Gamma] \cap \nu_{\theta}^{-1}[\{c\}]) \neq \varnothing$. Put $\Delta \triangleq \{\phi_c \mid c \in (\sigma[\Gamma]/\theta)\} \in \wp_{\omega}(\sigma[\Gamma])$. Consider any $\psi \in \sigma[\Gamma]$. Then, $\Delta \ni \phi_{\nu_\theta(\psi)} \equiv^{\omega}_{C} \psi$, in which case $\psi \in C(\Delta)$, and so $\sigma[\Gamma] \subseteq C(\Delta)$. In this way, $\sigma(\varphi) \in C(\Delta) = C'(\Delta)$, for $\Delta \in \wp_{\omega}(\mathrm{Fm}^{\omega}_{\Sigma})$, so, by (2.12), $\sigma(\varphi) \in \mathrm{Cn}^{\alpha}_{\mathsf{M}}(\Delta)$. Moreover, $g[\Delta] \subseteq g[\sigma[\Gamma]] = h[\Gamma] \subseteq D^{\mathcal{A}}$, and so $h(\varphi) = g(\sigma(\varphi)) \in D^{\mathcal{A}}$, as required. □

Corollary 3.28. *Let* M *be a finite class of finite* $\veebar$*-disjunctive* Σ*-matrices,* C *the logic of* M *and* C' *a* $\veebar$*-disjunctive extension of* C*. Then,* C' *is defined by* $\mathsf{S} \triangleq (\mathbf{S}_*(\mathsf{M}) \cap \mathrm{Mod}(C))$*, and so is finitary.*

Proof. Let $\mathcal{C}$ be the finitary Σ-calculus of all finitary Σ-rules satisfied in C', C'' the finitary sublogic of C' axiomatized by $\mathcal{C}$ and $\mathsf{S}' \triangleq (\mathbf{S}_*(\mathsf{M}) \cap \mathrm{Mod}(C'')) = (\mathbf{S}_*(\mathsf{M}) \cap \mathrm{Mod}(\mathcal{C}))$. Clearly, $C'' \subseteq \mathrm{Cn}^{\omega}_{\mathsf{S}'}$. Conversely, by Theorem 3.26 with $\mathcal{A} = \varnothing$, $\mathrm{Cn}^{\omega}_{\mathsf{S}'}$ is the extension of C relatively axiomatized by $\sigma_{+1}[\mathcal{C}] \veebar x_0$. On the other hand, by the structurality and $\veebar$-disjunctivity of C' as well as Lemma 3.25 with $X = \varnothing$, $(\sigma_{+1}[\mathcal{C}] \veebar x_0) \subseteq \mathcal{C}$. Moreover, C, being a finitary sublogic of C', is a sublogic of C'', in which case $C'' \supseteq \mathrm{Cn}^{\omega}_{\mathsf{S}'}$, and so C'' is defined by S', in which case C' is finitely-defined by S', and so is defined by S', by Lemma 3.27, in which case $C' = C''$, and so $\mathsf{S} = \mathsf{S}'$, as required. □

Proposition 3.29. *Let* M *be a [finite] class of [finite* $\veebar$*-disjunctive]* Σ*-matrices. Then,* $\mathbf{S}_*(\mathsf{M})$ *has no truth-empty member if[f] the logic of* M *has a theorem.*

Proof. The "if" part is by (2.13) and Remark 2.4. [Conversely, assume $\mathbf{S}_*(\mathsf{M})$ has no truth-empty member. Let $\overline{\mathcal{A}}$ be any enumeration of M. Consider any $i \in |\mathsf{M}| \in \omega$. Let $\bar{a}$ be any enumeration of $A_i \setminus D^{\mathcal{A}_i}$. Consider any $j \in (\mathrm{dom}\,\bar{a}) \in \omega$. Let $\mathfrak{B}$ be the subalgebra of $\mathcal{A}_i$ generated by $\{a_j\}$. Then, $(\mathcal{A}_i{\restriction}B) \in \mathbf{S}_*(\mathsf{M})$ is truth-non-empty, in which case there is some $\phi_j \in \mathrm{Fm}^1_{\Sigma}$ such that $\phi_j^{\mathfrak{A}_i}(a_j) \in D^{\mathcal{A}_i}$, and so $\psi_i \triangleq (\veebar\langle\bar{\phi}, x_0\rangle)$ is true in $\mathcal{A}_i$. In this way, $\veebar\langle\bar{\psi}, x_0\rangle$ is true in M, as required.] □

4. Super-classical matrices versus three-valued logics with subclassical negation

A Σ-matrix $\mathcal{A}$ is said to be $\sim$*-super-classical*, if $\mathcal{A}{\restriction}\{\sim\}$ has a $\sim$-classical submatrix, in which case $\mathcal{A}$ is both consistent and truth-non-empty, while, by (2.13), $\sim$ is a subclassical negation for the logic of $\mathcal{A}$, and so we have the "if" part of the following preliminary marking the framework of the present paper:

Theorem 4.1. *Let* $\mathcal{A}$ *be a* Σ*-matrix. [Suppose* $|A| \leqslant 3$*.] Then,* $\sim$ *is a subclassical negation for the logic of* $\mathcal{A}$ *if[f]* $\mathcal{A}$ *is* $\sim$*-super-classical.*

Proof. [Assume $\sim$ is a subclassical negation for the logic of $\mathcal{A}$. First, by (2.15) with $m = 1$ and $n = 0$, there is some $a \in D^{\mathcal{A}}$ such that $\sim^{\mathfrak{A}} a \notin D^{\mathcal{A}}$. Likewise, by (2.15) with $m = 0$ and $n = 1$, there is some $b \in (A \setminus D^{\mathcal{A}})$ such that $\sim^{\mathfrak{A}} b \in D^{\mathcal{A}}$, in which case $a \neq b$, and so $|A| \neq 1$. Then, if $|A| = 2$, we have $A = \{a, b\}$, in which case $\mathcal{A}$ is $\sim$-classical, and so $\sim$-super-classical. Now, assume $|A| = 3$.

Claim 4.2. *Let* $\mathcal{A}$ *be a three-valued* Σ*-matrix,* $\bar{a} \in A^2$ *and* $i \in 2$*. Suppose* $\sim$ *is a subclassical negation for the logic of* $\mathcal{A}$ *and, for each* $j \in 2$*,* $(a_j \in D^{\mathcal{A}}) \Leftrightarrow (\sim^{\mathfrak{A}} a_j \notin D^{\mathcal{A}}) \Leftrightarrow (a_{1-j} \notin D^{\mathcal{A}})$*. Then, either* $\sim^{\mathfrak{A}} a_i = a_{1-i}$ *or* $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a_i = a_i$*.*

Proof. By contradiction. For suppose both $\sim^{\mathfrak{A}} a_i \neq a_{1-i}$ and $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a_i \neq a_i$. Then, in case $a_i \in / \notin D^{\mathcal{A}}$, as $|A| = 3$, we have both $(D^{\mathcal{A}}/(A \setminus D^{\mathcal{A}})) = \{a_i\}$, in which case $\sim^{\mathfrak{A}} a_{1-i} = a_i$, and $((A \setminus D^{\mathcal{A}})/D^{\mathcal{A}}) = \{a_{1-i}, \sim^{\mathfrak{A}} a_i\}$, respectively. Consider the following exhaustive cases:

- $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a_i = a_{1-i}$.
  Then, $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a_i = a_i$. This contradicts to (2.15) with $(n/m) = 0$ and $(m/n) = 3$, respectively.
- $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a_i = \sim^{\mathfrak{A}} a_i$.
  Then, for each $c \in ((A \setminus D^{\mathcal{A}})/D^{\mathcal{A}})$, $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} c = \sim^{\mathfrak{A}} a_i \notin / \in D^{\mathcal{A}}$. This contradicts to (2.15) with $(n/m) = 3$ and $(m/n) = 0$, respectively.

Thus, in any case, we come to a contradiction, as required. □

Finally, consider the following exhaustive cases:

- both $\sim^{\mathfrak{A}} a = b$ and $\sim^{\mathfrak{A}} b = a$.
  Then, $\{a, b\}$ forms a subalgebra of $\mathfrak{A}\upharpoonright\{\sim\}$, $(\mathcal{A}\upharpoonright\{\sim\})\upharpoonright\{a, b\}$ being a $\sim$-classical submatrix of $\mathcal{A}\upharpoonright\{\sim\}$, as required.
- $\sim^{\mathfrak{A}} a \neq b$.
  Then, by Claim 4.2, $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a = a$, in which case $\{a, \sim^{\mathfrak{A}} a\}$ forms a subalgebra of $\mathfrak{A}\upharpoonright\{\sim\}$, $(\mathcal{A}\upharpoonright\{\sim\})\upharpoonright\{a, \sim^{\mathfrak{A}} a\}$ being a $\sim$-classical submatrix of $\mathcal{A}\upharpoonright\{\sim\}$, as required.
- $\sim^{\mathfrak{A}} b \neq a$.
  Then, by Claim 4.2, $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} b = b$, in which case $\{b, \sim^{\mathfrak{A}} b\}$ forms a subalgebra of $\mathfrak{A}\upharpoonright\{\sim\}$, $(\mathcal{A}\upharpoonright\{\sim\})\upharpoonright\{b, \sim^{\mathfrak{A}} b\}$ being a $\sim$-classical submatrix of $\mathcal{A}\upharpoonright\{\sim\}$, as required.] □

The following counterexample shows that the optional condition $|A| \leqslant 3$ is essential for the optional "only if" part of Theorem 4.1 to hold:

Example 4.3. Let $n \in \omega$ and $\mathcal{A}$ any Σ-matrix with $A \triangleq (n \cup (2 \times 2))$, $D^{\mathcal{A}} \triangleq \{\langle 1, 0\rangle, \langle 1, 1\rangle\}$, $\sim^{\mathfrak{A}}\langle i, j\rangle \triangleq \langle 1 - i, (1 - i + j) \mod 2\rangle$, for all $i, j \in 2$, and $\sim^{\mathfrak{A}} k \triangleq \langle 1, 0\rangle$, for all $k \in n$. Then, for any submatrix $\mathcal{B}$ of $\mathcal{A}\upharpoonright\{\sim\}$, we have $(2 \times 2) \subseteq B$, in which case $4 \leqslant |B|$, and so $\mathcal{A}$ is not $\sim$-super-classical, for $4 \nleqslant 2$. On the other hand, $(\mathcal{A}\upharpoonright\{\sim\})\upharpoonright(2 \times 2)$ is $\sim$-negative and consistent, in which case $\chi^{\mathcal{A}}\upharpoonright(2 \times 2)$ is a surjective strict homomorphism from it onto the $\sim$-classical $\{\sim\}$-matrix $\mathcal{C}$ with $C \triangleq 2$, $D^{\mathcal{C}} \triangleq \{1\}$ and $\sim^{\mathfrak{C}} i \triangleq (1 - i)$, for all $i \in 2$, and so, by (2.13), $\sim$ is a subclassical negation for the logic of $\mathcal{A}$. □

Let $\mathcal{A}$ be a fixed three-valued $\sim$-super-classical (in particular, both consistent and truth-non-empty) Σ-matrix and $\mathcal{B}$ a $\sim$-classical submatrix of $\mathcal{A}\upharpoonright\{\sim\}$. Then, as $4 \nleqslant 3$, $\mathcal{A}$ is either false-singular, in which case the unique non-distinguished value $0_{\mathcal{A}}$ of $\mathcal{A}$ is equal to $0_{\mathcal{B}}$, so $1^{\sim}_{\mathcal{A}} \triangleq \sim^{\mathfrak{A}} 0_{\mathcal{A}} = \sim^{\mathfrak{B}} 0_{\mathcal{B}} = 1_{\mathcal{B}}$, or truth-singular, in which case the unique distinguished value $1_{\mathcal{A}}$ of $\mathcal{A}$ is equal to $1_{\mathcal{B}}$, so $0^{\sim}_{\mathcal{A}} \triangleq \sim^{\mathfrak{A}} 1_{\mathcal{A}} = \sim^{\mathfrak{B}} 1_{\mathcal{B}} = 0_{\mathcal{B}}$. Thus, in case $\mathcal{A}$ is false-/truth-singular, $B = 2^{\sim}_{\mathcal{A}} \triangleq \{0^{/\sim}_{\mathcal{A}}, 1^{\sim/}_{\mathcal{A}}\}$ is uniquely determined by $\mathcal{A}$ and $\sim$, the unique element of $A \setminus 2^{\sim}_{\mathcal{A}}$ being denoted by $(\frac{1}{2})^{\sim}_{\mathcal{A}}$. (The indexes $_{\mathcal{A}}$ and, especially, $^{\sim}$ are often omitted, unless any confusion is possible.) Then, we have the partial ordering $\sqsubseteq \triangleq (\Delta_A \cup \{\langle \frac{1}{2}, i\rangle \mid i \in 2\})$ on A. An n-ary, where $n \in \omega$, operation on A is said to be *regular*, provided it is monotonic with respect to $\sqsubseteq$. Then, $\mathfrak{A}$ is said to be *regular*, whenever its primary operations are so, in which case secondary are so as well. Strict homomorphisms from $\mathcal{A}$ to itself retain both 0 and 1, in which case surjective ones retain $\frac{1}{2}$, and so:

$$\hom^{[\mathrm{S}]}(\mathcal{A}, \mathcal{A}) \supseteq [=]\{\Delta_A\}, \tag{4.1}$$

the inclusion being [not] allowed to be proper (cf. Example 5.8).

From now on, unless otherwise specified, C is supposed to be the logic of $\mathcal{A}$.

5. Non-classical three-valued logics with subclassical negation

Lemma 5.1. *Let $\mathcal{B}$ be a three-valued $\sim$-super-classical Σ-matrix. Then, following are equivalent:*

(i) *$\mathcal{B}$ is a strict surjective homomorphic counter-image of a $\sim$-classical Σ-matrix;*
(ii) *$\mathcal{B}$ is not simple;*
(iii) *$\mathcal{B}$ is not hereditarily simple;*
(iv) $\theta^{\mathcal{B}} \in \mathrm{Con}(\mathfrak{B})$.

Proof. First, (i)⇒(ii) is by Remark 2.5 and the fact that $3 \nleq 2$. Next, (iii) is a particular case of (ii). The converse is by the fact that any proper submatrix of $\mathcal{B}$, being either one-valued or $\wr$-classical, is simple. Further, (ii)⇒(iv) is by the following claim:

Claim 5.2. *Let $\mathcal{B}$ be a three-valued as well as both consistent and truth-non-empty Σ-matrix. Then, any non-diagonal congruence θ of it is equal to $\theta^{\mathcal{B}}$.*

Proof. First, we have $\theta \subseteq \theta^{\mathcal{B}}$. Conversely, consider any $\bar{a} \in \theta^{\mathcal{B}}$. Then, in case $a_0 = a_1$, we have $\bar{a} \in \Delta_B \subseteq \theta$. Otherwise, take any $\bar{b} \in (\theta \setminus \Delta_B) \neq \varnothing$, in which case $\bar{b} \in \theta^{\mathcal{B}}$, for $\theta \subseteq \theta^{\mathcal{B}}$. Then, as $|B| = 3 \ngeq 4$, there are some $i, j \in 2$ such that $a_i = b_j$. Hence, if a_{1-i} was not equal to b_{1-j}, then we would have both $|\{a_i, a_{1-i}, b_{1-j}\}| = 3 = |B|$, in which case we would get $\{a_i, a_{1-i}, b_{1-j}\} = B$, and $\chi^{\mathcal{B}}(b_{1-j}) = \chi^{\mathcal{B}}(b_j) = \chi^{\mathcal{B}}(a_i) = \chi^{\mathcal{B}}(a_{1-i})$, and so $\mathcal{B}$ would be either truth-empty or inconsistent. Therefore, both $a_{1-i} = b_{1-j}$ and $a_i = b_j$. Thus, since θ is symmetric, we eventually get $\bar{a} \in \theta$, for $\bar{b} \in \theta$, as required. □

Finally, assume (iv) holds. Then, $\theta \triangleq \theta^{\mathcal{B}}$, including itself, is a congruence of $\mathcal{B}$, in which case $\nu_\theta \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{B}, \mathcal{B}/\theta)$, while $\mathcal{B}/\theta$ is $\sim$-classical, and so (i) holds. □

Set $h_{+/2} : 2^2 \to (3 \div 2), \langle i, j\rangle \mapsto \frac{i+j}{2}$.

Theorem 5.3. *The following are equivalent:*

(i) *C is $\sim$-classical;*
(ii) *either $\mathcal{A}$ is a strict surjective homomorphic counter-image of a $\sim$-classical Σ-matrix or $\mathcal{A}$ is a strict surjective homomorphic image of a submatrix of a direct power of a $\sim$-classical Σ-matrix;*
(iii) *either $\mathcal{A}$ is a strict surjective homomorphic counter-image of a $\sim$-classical Σ-matrix or $\mathcal{A}$ is a strict surjective homomorphic image of the direct square of a $\sim$-classical Σ-matrix;*
(iv) *either $\mathcal{A}$ is not simple or both $2_{\mathcal{A}}$ forms a subalgebra of $\mathfrak{A}$ and $\mathcal{A}$ is a strict surjective homomorphic image of $(\mathcal{A}{\restriction}2_{\mathcal{A}})^2$;*
(v) *either $\theta^{\mathcal{A}} \in \mathrm{Con}(\mathfrak{A})$ or both $2_{\mathcal{A}}$ forms a subalgebra of $\mathfrak{A}$, $\mathcal{A}$ is truth-singular and $h_{+/2} \in \hom((\mathfrak{A}{\restriction}2_{\mathcal{A}})^2, \mathfrak{A})$.*

In particular, [providing $\mathcal{A}$ is false-singular] $\mathcal{A}$ is (hereditarily) simple if[f] C is non-$\sim$-classical.

Proof. We use Lemma 5.1 tacitly. First, (ii/iii/iv) is a particular case of (iii/iv/v), respectively. Next, (iv)⇒(i) is by (2.13). Further, (i)⇒(ii) is by Lemmas 2.7, 2.8 and Remark 2.5.

Now, let $\mathcal{B}$ be a $\sim$-classical Σ-matrix, I a set, $\mathcal{D}$ a submatrix of $\mathcal{B}^I$ and $h \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{D}, \mathcal{A})$, in which case $\mathcal{D}$ is both consistent and truth-non-empty, for $\mathcal{A}$ is so, and so $I \neq \varnothing$, while, as $\mathcal{B}$ is truth-singular, $a \triangleq (I \times \{1_{\mathcal{B}}\}) \in D^{\mathcal{B}}$, whereas, for this reason, $D \ni b \triangleq \sim^{\mathfrak{D}} a = (I \times \{1_{\mathcal{B}}\}) \notin D^{\mathcal{D}}$, for $I \neq \varnothing$. Then, $\sim^{\mathfrak{D}} b = a$, in which case $h(a/b) = (1/0)_{\mathcal{A}}$, and so there is some $c \in (D \setminus \{a,b\})$ such that $h(c) = (\frac{1}{2})_{\mathcal{A}}$. In this way, $I \neq J \triangleq \{i \in I \mid \pi_i(c) = 1_{\mathcal{B}}\} \neq \varnothing$. Given any $\bar{a} \in B^2$, set $(a_0 \wr a_1) \triangleq ((J \times \{a_0\}) \cup ((I \setminus J) \times \{a_1\}))$. Then, $D \ni a = (1_{\mathcal{B}} \wr 1_{\mathcal{B}})$ and $D \ni b = (0_{\mathcal{B}} \wr 0_{\mathcal{B}})$ as well as $D \ni c = (1_{\mathcal{B}} \wr 0_{\mathcal{B}})$, in which case $D \ni \sim^{\mathfrak{D}} c = (0_{\mathcal{B}} \wr 1_{\mathcal{B}})$, and so $e \triangleq \{\langle\langle x,y\rangle,(x \wr y)\rangle \mid x,y \in B\}$ is an embedding of $\mathcal{B}^2$ into D such that $\{a,b,c\} \subseteq (\mathrm{img}\, e)$. Hence, since $h[\{a,b,c\}] = A$, we conclude that $(h \circ e) \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{B}^2,\mathcal{A})$. Thus, (ii)⇒(iii) holds.

Likewise, let $\mathcal{B}$ be a $\sim$-classical Σ-matrix and $g \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{B}^2,\mathcal{A})$. Then, $e' \triangleq (\Delta_B \times \Delta_B)$ is an embedding of $\mathcal{B}$ into $\mathcal{B}^2$, in which case, by Remark 2.5, $g' \triangleq (g \circ e')$ is an embedding of $\mathcal{B}$ into $\mathcal{A}$, and so $E \triangleq (\mathrm{img}\, g')$ forms a two-element subalgebra of $\mathfrak{A}$, g' being an isomorphism from $\mathcal{B}$ onto $\mathcal{E} \triangleq (\mathcal{A} \restriction E)$. Therefore, as $\mathfrak{A} \restriction \{\sim\}$ has no two-element subalgebra other than that with carrier $2_{\mathcal{A}}$, $E = 2_{\mathcal{A}}$. And what is more, $(g \circ ((g'^{-1} \circ (\pi_0 \restriction E^2)) \times (g'^{-1} \circ (\pi_0 \restriction E^2)))) \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{E}^2,\mathcal{A})$. Thus, (iii)⇒(iv) holds.

Finally, assume (iv) holds, while $\mathcal{A}$ is simple. Then, $\mathcal{A}$ is truth-singular, for $\mathcal{F} \triangleq (\mathcal{A} \restriction 2_{\mathcal{A}})$ is so. Let $f \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{F}^2,\mathcal{A})$. Then, $\sim^{\mathfrak{A}^2}\langle(0/1)_{\mathcal{A}},(0/1)_{\mathcal{A}}\rangle = \langle(1/0)_{\mathcal{A}},(1/0)_{\mathcal{A}}\rangle \in / \notin D^{\mathcal{F}^2}$. Hence, $f(\langle(0/1)_{\mathcal{A}},(0/1)_{\mathcal{A}}\rangle) = (0/1)_{\mathcal{A}}$. Moreover, $\sim^{\mathfrak{A}^2}\langle(0/1)_{\mathcal{A}},(1/0)_{\mathcal{A}}\rangle = \langle(1/0)_{\mathcal{A}},(0/1)_{\mathcal{A}}\rangle \notin D^{\mathcal{F}^2}$. Hence, $f(\langle(0/1)_{\mathcal{A}},(1/0)_{\mathcal{A}}\rangle) = (\frac{1}{2})_{\mathcal{A}}$, so (v) holds. □

The simplicity of $\mathcal{A}$ is not, generally speaking, sufficient for C's being non-$\sim$-classical, even if this is conjuctive, as it follows from:

Example 5.4. Let $\Sigma \triangleq \{\wedge,\sim\}$, $D^{\mathcal{A}} \triangleq \{1\}$, $\sim^{\mathfrak{A}} a \triangleq (1-a)$, for all $a \in A$, and

$$(a \wedge^{\mathfrak{A}} b) \triangleq \begin{cases} a & \text{if } a = 1 = b, \\ 0 & \text{otherwise.} \end{cases}$$

for all $a,b \in A$. Then, $\mathcal{A}$ is both truth-singular and $\wedge$-conjunctive, while $\langle 0,\frac{1}{2}\rangle \in \theta^{\mathcal{A}} \not\ni \langle 1,\frac{1}{2}\rangle = \langle \sim^{\mathfrak{A}} 0, \sim^{\mathfrak{A}} \frac{1}{2}\rangle$, in which case $\theta^{\mathcal{A}} \notin \mathrm{Con}(\mathfrak{A})$, and so, by Lemma 5.1, is simple. On the other hand, 2 forms a subalgebra of $\mathfrak{A}$, while $h_{+/2} \in \hom((\mathfrak{A}\restriction 2)^2,\mathfrak{A})$. Hence, by Theorem 5.3, C is $\sim$-classical. □

5.1. **The uniqueness of defining super-classical matrix.**

Lemma 5.5. *Let $\mathcal{B}$ be a $\sim$-paraconsistent $\sim$-super-classical Σ-matrix. Suppose $\mathcal{B}$ is a model of C (in particular, C is defined by $\mathcal{B}$). Then, $\mathcal{A}$ is embeddable into $\mathcal{B}$.*

Proof. In that case, C (viz., $\mathcal{A}$) is $\sim$-paraconsistent too, and so both $\mathcal{A}$ and $\mathcal{B}$ are simple, by Theorem 5.3, and weakly $\sim$-negative. Moreover, $\mathcal{B}$ is a finite $\sim$-paraconsistent model of C. Therefore, by Lemmas 2.7, 2.8 and Remark 2.5, there are some non-empty set I, some I-tuple $\overline{\mathcal{C}}$ constituted by submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$ and some $g \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{D},\mathcal{B})$, in which case $\mathcal{D}$ is both weakly $\sim$-negative and, by (2.13), is $\sim$-paraconsistent, for $\mathcal{B}$ is so, and so there are some $a \in D^{\mathcal{D}}$ such that $\sim^{\mathfrak{D}} a \in D^{\mathcal{D}}$ and some $b \in (D \setminus D^{\mathcal{D}})$, in which case $c \triangleq \sim^{\mathfrak{D}} b \in D^{\mathcal{D}} \subseteq \{(\frac{1}{2})_{\mathcal{A}}, 1_{\mathcal{A}}\}^I$, for $\mathcal{D}$ is weakly $\sim$-negative. Then, $D \ni a = (I \times \{(\frac{1}{2})_{\mathcal{A}}\})$. Consider the following complementary cases:

- $\{(\frac{1}{2})_{\mathcal{A}}\}$ forms a subalgebra of $\mathfrak{A}$,
  in which case $\sim^{\mathfrak{A}}(\frac{1}{2})_{\mathcal{A}} = (\frac{1}{2})_{\mathcal{A}}$, and so $\sim^{\mathfrak{D}} c = b \notin D^{\mathcal{B}}$. Hence, $J \triangleq \{i \in I \mid \pi_i(c) = 1_{\mathcal{A}}\} \neq \varnothing$. Given any $\bar{a} \in A^2$, set $(a_0 \wr a_1) \triangleq ((J \times \{a_0\}) \cup ((I \setminus J) \times \{a_1\})) \in A^I$. In this way, $D \ni a = ((\frac{1}{2})_{\mathcal{A}} \wr (\frac{1}{2})_{\mathcal{A}})$, $D \ni c = (1_{\mathcal{A}} \wr (\frac{1}{2})_{\mathcal{A}})$ and $D \ni b = (0_{\mathcal{A}} \wr (\frac{1}{2})_{\mathcal{A}})$. Then, as $\{(\frac{1}{2})_{\mathcal{A}}\}$ forms a subalgebra of $\mathfrak{A}$, while $J \neq \varnothing$, $f \triangleq \{\langle d,(d \wr (\frac{1}{2})_{\mathcal{A}})\rangle \mid d \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$.

- $\{(\frac{1}{2})_{\mathcal{A}}\}$ does not form a subalgebra of $\mathfrak{A}$.
  Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}((\frac{1}{2})_{\mathcal{A}}) \neq (\frac{1}{2})_{\mathcal{A}}$, in which case $\{(\frac{1}{2})_{\mathcal{A}}, \varphi^{\mathfrak{A}}((\frac{1}{2})_{\mathcal{A}}), \sim^{\mathfrak{A}}\varphi^{\mathfrak{A}}((\frac{1}{2})_{\mathcal{A}})\} = A$, and so $D \supseteq \{a, \varphi^{\mathfrak{D}}(a), \sim^{\mathfrak{D}}\varphi^{\mathfrak{D}}(a)\} = \{I \times \{d\} \mid d \in A\}$. Therefore, as $I \neq \varnothing$, $f \triangleq \{\langle d, I \times \{d\}\rangle \mid d \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$.

Thus, $h \triangleq (g \circ f) \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$, and so Remark 2.5 completes the argument. □

Theorem 5.6. *Let $\mathcal{B}$ be a $\sim$-super-classical Σ-matrix. Suppose C is defined by $\mathcal{B}$ and is not $\sim$-classical. Then, $\mathcal{B}$ is isomorphic to $\mathcal{A}$.*

Proof. In that case, both $\mathcal{A}$ and $\mathcal{B}$ are simple, by Theorem 5.3. In particular, by Lemmas 2.7, 2.8 and Remark 2.5, $\mathcal{A}$ is truth-singular iff $\mathcal{B}$ is so, in which case $\mathcal{A}$ is false-singular iff $\mathcal{B}$ is so, for $\mathcal{A}/\mathcal{B}$ is false-singular iff it is not truth-singular. By contradiction, we are going to prove that $\hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B}) \neq \varnothing$. For suppose $\hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B}) = \varnothing$. Then, by Lemma 5.5, C (viz., $\mathcal{A}/\mathcal{B}$) is non-$\sim$-paraconsistent, in which case $\{(\frac{1}{2})_{\mathcal{A}/\mathcal{B}}, \sim^{\mathfrak{A}/\mathfrak{B}}(\frac{1}{2})_{\mathcal{A}/\mathcal{B}}\} \nsubseteq D^{\mathcal{A}/\mathcal{B}}$, for $\mathcal{A}/\mathcal{B}$ is consistent. Moreover, by Lemmas 2.7, 2.8 and Remark 2.5, there are some non-empty set I, some I-tuple $\overline{\mathcal{C}}$ constituted by submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{B})$. Given any $a \in A$, set $(I : a) \triangleq (I \times \{a\}) \in A^I$. Consider the following complementary cases:

- $\mathcal{A}$ is truth-singular,
  in which case $\mathcal{B}$ is so. Moreover, $\mathcal{D}$ is truth-non-empty, for $\mathcal{B}$ is so. Take any $a \in D^{\mathcal{D}}$, in which case $D \ni a = (I : 1_{\mathcal{A}})$, while $g(a) \in D^{\mathcal{B}}$, and so $g(a) = 1_{\mathcal{B}}$. In particular, $D \ni b \triangleq \sim^{\mathfrak{D}} a = (I : 0^{\sim}_{\mathcal{A}})$, and so $g(b) = 0^{\sim}_{\mathcal{B}}$.
- $\mathcal{A}$ is false-singular,
  in which case $\mathcal{B}$ is so. Moreover, $\mathcal{D}$ is consistent, for $\mathcal{B}$ is so. Take any $b \in (D \setminus D^{\mathcal{D}})$, in which case, by the following claim, $D \ni b = (I : 0_{\mathcal{A}})$, while $g(b) \notin D^{\mathcal{B}}$, and so $g(b) = 0_{\mathcal{B}}$:

  Claim 5.7. *Let $\mathcal{B}$, I, $\mathcal{D}$ and g be as above. Suppose $\mathcal{A}$ is false-singular and not $\sim$-paraconsistent. Then, every $d \in (D \setminus D^{\mathcal{D}})$ is equal to $I : 0_{\mathcal{A}}$.*

  Proof. Then, $g(d) \in (B \setminus D^{\mathcal{B}})$, in which case $g(d) = 0_{\mathcal{B}}$, and so $g(\sim^{\mathfrak{D}} d) = 1^{\sim}_{\mathcal{B}} \in D^{\mathcal{B}}$. Hence, $\sim^{\mathfrak{D}} d \in D^{\mathcal{B}}$. Moreover, as $\mathcal{A}$ is false-singular, we have $(\frac{1}{2})_{\mathcal{A}} \in D^{\mathcal{A}}$, in which case $\sim^{\mathfrak{A}}(\frac{1}{2})_{\mathcal{A}} \notin D^{\mathcal{A}}$, for $\mathcal{A}$ is both consistent and non-$\sim$-paraconsistent, and so $\sim^{\mathfrak{A}} c \notin D^{\mathcal{A}}$, for all $c \in D^{\mathcal{A}}$. In this way, $d = (I : 0_{\mathcal{A}})$, as required. □

  In particular, $D \ni a \triangleq \sim^{\mathfrak{D}} b = (I : 1^{\sim}_{\mathcal{A}})$, and so $g(a) = 1^{\sim}_{\mathcal{B}}$.

Thus, anyway, $a = (I : 1_{\mathcal{A}}) \in D \ni b = (I : 0_{\mathcal{A}})$, while $g(a) = 1_{\mathcal{B}}$, whereas $g(b) = 0_{\mathcal{B}}$. Consider the following complementary cases:

- $2_{\mathcal{A}}$ does not form a subalgebra of $\mathfrak{A}$,
  in which case there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(1_{\mathcal{A}}, 0_{\mathcal{A}}) = (\frac{1}{2})_{\mathcal{A}}$, and so $D \in \varphi^{\mathfrak{D}}(a, b) = (I : (\frac{1}{2})_{\mathcal{A}})$. In this way, as $I \neq \varnothing$, $e \triangleq \{\langle x, I : x\rangle \mid x \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case $(g \circ e) \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$, and so this contradicts to the assumption that $\hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B}) = \varnothing$.
- $2_{\mathcal{A}}$ forms a subalgebra of $\mathfrak{A}$,
  in which case $\mathcal{E} \triangleq (\mathcal{A} \restriction 2_{\mathcal{A}})$ is $\sim$-classical, while $a, b \in E^I$. Then, $(\frac{1}{2})_{\mathcal{B}} \in B = g[D]$, in which case there is some $c \in D$ such that $g(c) = (\frac{1}{2})_{\mathcal{B}}$. Let $J \triangleq \{i \in I \mid \pi_i(c) = (\frac{1}{2})_{\mathcal{A}}\}$, in which case $\pi_i(c) \in E$, for all $i \in (I \setminus J)$. Let $\mathfrak{F}$ be the subalgebra of $\mathfrak{D}$ generated by $\{a, b, c\}$ and $\mathcal{F} \triangleq (\mathcal{D} \restriction F)$, in which case $f \triangleq (g \restriction F) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{F}, \mathcal{B})$, for $g[\{a, b, c\}] = B$. In particular, if J was empty, then c would be in E^I, in which case $\mathcal{F}$ would be a submatrix of

$\mathcal{E}^I$, and so, by (2.13), C would be $\sim$-classical. Therefore, $J \neq \varnothing$. Take any $j \in J$. Let us prove, by contradiction, that $(\pi_j \restriction F) \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{F}, \mathcal{A})$. For suppose $(\pi_j \restriction F) \notin \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{F}, \mathcal{A})$. Then, as $(\pi_j \restriction F) \in \hom^{\mathrm{S}}(\mathcal{F}, \mathcal{A})$, there is some $d \in (F \setminus D^{\mathcal{F}})$ such that $\pi_j(d) \in D^{\mathcal{A}}$. Consider the following complementary subcases:

– $\mathcal{A}$ is false-singular.
Then, by Claim 5.7, $D^{\mathcal{A}} \ni \pi_j(d) = 0_{\mathcal{A}}$.

– $\mathcal{A}$ is truth-singular.
Then, $\pi_j(d) = 1_{\mathcal{A}} = \pi_i(d)$, for all $i \in J$, because $\pi_j(e) = \pi_i(e)$, for all $e \in \{a, b, c\}$, and so for all $e \in F \ni d$, in which case $d \in E^I \supseteq \{a, b\}$, and so the subalgebra $\mathfrak{G}$ of $\mathfrak{F}$ generated by $\{a, b, d\}$ is a subalgebra of $\mathfrak{E}^I$. Moreover, $\pi_j(\sim^{\mathfrak{F}} d) = 0_{\mathcal{A}} \notin D^{\mathcal{A}}$, in which case $(\{d, \sim^{\mathfrak{F}} d\} \cap D^{\mathcal{F}}) = \varnothing$, and so $(\{f(d), \sim^{\mathfrak{B}} f(d)\} \cap D^{\mathcal{B}}) = \varnothing$. Hence, $f(d) = (\frac{1}{2})_{\mathcal{B}}$, in which case $f[\{a, b, d\}] = B$, and so $(f \restriction G) \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{F} \restriction G, \mathcal{B})$. In this way, by (2.13), C is $\sim$-classical.

Thus, anyway, we come to a contradiction. Therefore, $(\pi_j \restriction F) \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{F}, \mathcal{A})$. Hence, by Remark 2.5 and Lemma 2.7, $\mathcal{A}$ is isomorphic to $\mathcal{B}$. This contradicts to the assumption that $\hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B}) = \varnothing$.

Thus, in any case, we come to a contradiction. Therefore, there is some $h' \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$. Likewise, by symmetry, there is some $g' \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A})$. Then, $((g' \circ h')/(h' \circ g')) \in \hom_{\mathrm{S}}(\mathcal{A}/\mathcal{B}, \mathcal{A}/\mathcal{B})$ is injective, in view of Remark 2.5, and so bijective, for $|A/B| = 3$ is finite. In this way, (4.1) completes the argument. □

In view of Theorems 4.1 and 5.6, any [non-$\sim$-classical] three-valued Σ-logic with subclassical negation $\sim$ is defined by a [unique (either up to isomorphism or when dealing with merely *canonical* three-valued $\sim$-super-classical Σ-matrices, i.e., those of the form $\mathcal{A}'$ with $A' = (3 \div 2)$ and $a_{\mathcal{A}'} = a$, for all $a \in A'$, in which case isomorphic ones are equal, by (4.1) applied to their common $\sim$-reduct)] three-valued $\sim$-super-classical Σ-matrix [the unique canonical one being said to be *characteristic for|of* the logic]. On the other hand, such is not the case for $\sim$-classical (even both conjunctive and disjunctive) ones, in view of Theorem 4.1 and the following counterexample:

Example 5.8. Let $\Sigma \triangleq (\Sigma_+ \cup \{\sim\})$ and $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{E}$ the $\wedge$-conjunctive $\veebar$-disjunctive Σ-matrices with $(\mathfrak{B} \restriction \Sigma_+) \triangleq \mathfrak{D}_3$, $(\mathfrak{D} \restriction \Sigma_+) \triangleq \mathfrak{D}_3$, $(\mathfrak{E} \restriction \Sigma_+) \triangleq \mathfrak{D}_2$, $\sim^{\mathfrak{B}} i \triangleq (1 - \min(1, 2 \cdot i))$ and $\sim^{\mathfrak{D}} i \triangleq (1 - \max(0, (2 \cdot i) - 1)))$, for all $i \in (3 \div 2)$, $\sim^{\mathfrak{E}} i \triangleq (1 - i)$, for all $i \in 2$, $D^{\mathcal{B}} \triangleq \{1, \frac{1}{2}\}$, $D^{\mathcal{D}} \triangleq \{1\}$ and $D^{\mathcal{E}} \triangleq \{1\}$. Then, both $\mathcal{B}$ and $\mathcal{D}$ are three-valued and $\sim$-super-classical, while $\mathcal{C}$ is $\sim$-classical. And what is more, $\chi^{\mathcal{B}/\mathcal{D}} \in \hom^{\mathrm{S}}(\mathcal{B}/\mathcal{D}, \mathcal{E})$, in which case, by (2.13), $\mathcal{B}$ and $\mathcal{D}$ define the same $\sim$-classical Σ-logic of $\mathcal{E}$. However, $\mathcal{B}$, being false-singular, is not isomorphic to $\mathcal{D}$, not being so. Moreover, $\mathcal{E}$ is a submatrix of $\mathcal{B}/\mathcal{D}$, in which case $h \triangleq (\Delta_2 \circ \chi^{\mathcal{B}/\mathcal{D}})$ is a non-diagonal (for $h(\frac{1}{2}) = (1/0) \neq \frac{1}{2}$) strict homomorphism from $\mathcal{B}/\mathcal{D}$ to itself, and so the "[]"-non-optional inclusion in (4.1) may be proper. □

Corollary 5.9. *Let $\Sigma' \supseteq \Sigma$ be a signature and C' a three-valued Σ'-expansion of C. Then, C' is defined by a Σ'-expansion of $\mathcal{A}$.*

Proof. In that case, $\sim$ is a subclassical negation for C'. Hence, by Theorem 4.1, C' is defined by a $\sim$-super-classical Σ'-matrix $\mathcal{A}'$, in which case C is defined by the $\sim$-super-classical Σ-matrix $\mathcal{A}' \restriction \Sigma$, and so, by Theorem 5.6, there is some isomorphism e from $(\mathcal{A}' \restriction \Sigma)$ onto $\mathcal{A}$, in which case it is an isomorphism from $\mathcal{A}'$ onto the Σ'-expansion $\mathcal{A}'' \triangleq \langle e[\mathfrak{A}'], e[D^{\mathcal{A}'}] \rangle$ of $\mathcal{A}$, and so, by (2.13), C' is defined by $\mathcal{A}''$. □

6. Classical extensions

Lemma 6.1. *Let I be a set and $\mathcal{B}$ a consistent submatrix of $\mathcal{A}^I$, in which case $I \neq \varnothing$. Suppose $a \triangleq (I \times \{0\}) \in B$, that is, $b \triangleq (I \times \{1\}) \in B$ (in particular, $\mathcal{A}$ is truth-singular, while $\mathcal{B}$ is truth-non-empty), while $\mathcal{A}$ is not a model of the logic of $\mathcal{B}$. Then, the following hold:*

(i) 2 *forms a subalgebra of* $\mathfrak{A}$;
(ii) $\mathcal{A}{\restriction}2$ *is embeddable into* $\mathcal{B}$.

Proof. (i) By contradiction. For suppose 2 does not form a subalgebra of $\mathfrak{A}$. Then, there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(0,1) = \frac{1}{2}$, in which case $B \ni c \triangleq \varphi^{\mathfrak{B}}(a,b) = (I \times \{\frac{1}{2}\})$, and so $\{\langle d, I \times \{d\}\rangle \mid d \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{B}$. In view of (2.13), this contradicts to the assumption that $\mathcal{A}$ is not a model of the logic of $\mathcal{B}$.
(ii) As $I \neq \varnothing$, by (i), $\{\langle d, I \times \{d\}\rangle \mid d \in 2\}$ is an embedding of $\mathcal{A}{\restriction}2$ into $\mathcal{B}$, as required. □

A $(2[+1])$-*ary* [$\frac{1}{2}$-*relative*] *(classical) semi-conjunction for* $\mathcal{A}$ is any $\varphi \in \mathrm{Fm}^{2[+1]}_\Sigma$ such that both $\varphi^{\mathfrak{A}}(0,1[,\frac{1}{2}]) = 0$ and $\varphi^{\mathfrak{A}}(1,0[,\frac{1}{2}]) \in \{0[,\frac{1}{2}]\}$. (Clearly, any binary semi-conjunction for $\mathcal{A}$ is a ternary $\frac{1}{2}$-relative one.) Next, $\mathcal{A}$ is said to *satisfy generation condition (GC)*, provided either $\langle 0,0\rangle$ or $\langle \frac{1}{2},0\rangle$ or $\langle 0,\frac{1}{2}\rangle$ belongs to the carrier of the subalgebra of $\mathfrak{A}^2$ generated by $\{\langle 1,\frac{1}{2}\rangle\}$.

Lemma 6.2 (Key "False-singular" Lemma)**.** *Let I be a set and $\mathcal{B}$ a consistent submatrix of $\mathcal{A}^I$, in which case $I \neq \varnothing$. Suppose $\mathcal{A}$ is false-singular and not a model of the logic of $\mathcal{B}$, while $\mathcal{B}$ is not $\sim$-paraconsistent, whereas either $\mathcal{B}$ is $\sim$-negative or both either $\mathcal{A}$ has a binary semi-conjunction or both $\mathcal{B}$ is truth-non-empty and $\mathcal{A}$ satisfies GC, and either 2 forms a subalgebra of $\mathfrak{A}$ or $L_4 \triangleq (A^2 \setminus (2^2 \cup \{\frac{1}{2}\}^2))$ forms a subalgebra of $\mathfrak{A}^2$. Then, the following hold:*

(i) *if* 2 *forms a subalgebra of* $\mathfrak{A}$, *then* $\mathcal{A}{\restriction}2$ *is embeddable into* $\mathcal{B}$;
(ii) *if* 2 *does not form a subalgebra of* $\mathfrak{A}$, *then* L_4 *forms a subalgebra of* $\mathfrak{A}^2$, *while* $(\mathcal{A}^2{\restriction}L_4)$ *is embeddable into* $\mathcal{B}$.

Proof. We start from proving that there is some non-empty $J \subseteq I$ such that $(1 \wr \frac{1}{2}) \in B$, where, for every $\bar{a} \in A^2$, we set $(a_0 \wr a_1) \triangleq ((J \times \{a_0\}) \cup ((I \setminus J) \times \{a_1\})) \in A^I$. Take any $a \in (B \setminus D^{\mathcal{B}}) \neq \varnothing$, for $\mathcal{B}$ is consistent. Consider the following exhaustive cases:

- $\mathcal{B}$ is $\sim$-negative.
  Then, $b \triangleq \sim^{\mathfrak{B}} a \in D^{\mathcal{B}} \subseteq \{\frac{1}{2}, 1\}^I$, in which case $B \ni c \triangleq \sim^{\mathfrak{B}} b \notin D^{\mathcal{B}}$, and so $J \triangleq \{i \in I \mid \pi_i(b) = 1\} \neq \varnothing$. In this way, $B \ni b = (1 \wr \frac{1}{2})$.
- $\mathcal{A}$ has a binary semi-conjunction φ.
  Let $K \triangleq \{i \in I \mid \pi_i(a) = 1\}$, $L \triangleq \{i \in I \mid \pi_i(a) = 0\} \neq \varnothing$, for $a \notin D^{\mathcal{B}}$. Given any $\bar{a} \in A^3$, we set $(a_0 \wr a_1 \wr a_2) \triangleq ((K \times \{a_0\}) \cup (L \times \{a_1\}) \cup ((I \setminus (K \cup L)) \times \{a_2\})) \in A^I$. In this way, $B \ni a = (1 \wr 0 \wr \frac{1}{2})$. Consider the following exhaustive subcases:
  - $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$.
    Then, $B \ni b \triangleq \sim^{\mathfrak{A}} a = (0 \wr 1 \wr \frac{1}{2})$. Let $x \triangleq \varphi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2}) \in A$. Consider the following exhaustive subsubcases:
    - $x = \frac{1}{2}$.
      Then, $B \ni c \triangleq \varphi^{\mathfrak{B}}(a,b) = (0 \wr 0 \wr \frac{1}{2})$. Put $J \triangleq (K \cup L) \neq \varnothing$, for $L \neq \varnothing$. In this way, $(1 \wr \frac{1}{2}) = \sim^{\mathfrak{B}} c \in B$.

    * $x = 0$.
      Then, $B \ni c \triangleq \varphi^{\mathfrak{B}}(a,b) = (0 \wr 0 \wr 0)$. Put $J \triangleq I \neq \varnothing$. In this way, $(1 \wr \frac{1}{2}) = \sim^{\mathfrak{B}} c \in B$.
    * $x = 1$.
      Then, $B \ni c \triangleq \varphi^{\mathfrak{B}}(a,b) = (0 \wr 0 \wr 1)$, and so $B \ni \sim^{\mathfrak{B}} c = (1 \wr 1 \wr 0)$. Put $J \triangleq I \neq \varnothing$. Then, $(1 \wr \frac{1}{2}) = \sim^{\mathfrak{B}} \varphi^{\mathfrak{B}}(c, \sim^{\mathfrak{B}} c) \in B$.
  - $\sim^{\mathfrak{A}} \frac{1}{2} = 1$.
    Then, $B \ni b \triangleq \sim^{\mathfrak{A}} a = (0 \wr 1 \wr 1)$, and so $B \ni \sim^{\mathfrak{B}} b = (1 \wr 0 \wr 0)$. Put $J \triangleq I \neq \varnothing$. Then, $(1 \wr \frac{1}{2}) = \sim^{\mathfrak{B}} \varphi^{\mathfrak{B}}(b, \sim^{\mathfrak{B}} b) \in B$.
  - $\sim^{\mathfrak{A}} \frac{1}{2} = 0$.
    Then, $B \ni b \triangleq \sim^{\mathfrak{A}} a = (0 \wr 1 \wr 0)$, and so $B \ni \sim^{\mathfrak{B}} b = (1 \wr 0 \wr 1)$. Put $J \triangleq I \neq \varnothing$. Then, $(1 \wr \frac{1}{2}) = \sim^{\mathfrak{B}} \varphi^{\mathfrak{B}}(b, \sim^{\mathfrak{B}} b) \in B$.
- $\mathcal{B}$ is truth-non-empty.
  Take any $d \in D^{\mathcal{B}} \subseteq (D^{\mathcal{A}})^I$. Let $J \triangleq \{i \in I \mid \pi_i(d) = 1\}$. Then, as $\mathcal{B}$ is not $\sim$-paraconsistent, we have $J \neq \varnothing$, for, otherwise, (2.10) would not be true in $\mathcal{B}$ under $[x_0/d, x_1/a]$. In this way, $(1 \wr \frac{1}{2}) = d \in B$.

Further, we prove:

Claim 6.3. *Suppose $\sim^{\mathfrak{A}} \frac{1}{2} \neq \frac{1}{2}$. Then, L_4 does not form a subalgebra of $\mathfrak{A}^2$ and, providing both I, $\mathcal{B}$, J and $(1 \wr \frac{1}{2}) \in B$ are as above, $(I \times \{1\}) \in B$.*

Proof. First, in case $\sim^{\mathfrak{A}} \frac{1}{2} = (0/1)$, we have, respectively, $\sim^{\mathfrak{A}^2} \langle \frac{1}{2}, 1/0 \rangle = \langle 0/1, 0/1 \rangle \notin L_4$, and so $L_4 \ni \langle \frac{1}{2}, 1/0 \rangle$ does not form a subalgebra of $\mathfrak{A}^2$. Finally, consider the following complementary cases:

- $\sim^{\mathfrak{A}} \frac{1}{2} = 0$.
  Then, $(I \times \{1\}) = \sim^{\mathfrak{B}} \sim^{\mathfrak{B}} (1 \wr \frac{1}{2}) \in B$.
- $\sim^{\mathfrak{A}} \frac{1}{2} = 1$.
  Then, consider the following exhaustive subcases:
  - $\mathcal{B}$ is $\sim$-negative.
    Then, $(1 \wr \frac{1}{2}) \in D^{\mathcal{B}}$, in which case $(1 \wr 0) = \sim^{\mathfrak{B}} \sim^{\mathfrak{B}} (1 \wr \frac{1}{2}) \in D^{\mathcal{B}}$, and so $J = I$. In this way, $(I \times \{1\}) = (1 \wr \frac{1}{2}) \in B$, as required.
  - $\mathcal{A}$ has a binary semi-conjunction φ.
    Then, $b \triangleq (0 \wr 1) = \sim^{\mathfrak{B}} (1 \wr \frac{1}{2}) \in B$, and so $B \ni \sim^{\mathfrak{B}} b = (1 \wr 0)$. In this way, $(I \times \{1\}) = \sim^{\mathfrak{B}} \varphi^{\mathfrak{B}}(b, \sim^{\mathfrak{B}} b) \in B$, as required.
  - $\mathcal{A}$ satisfies GC.
    Then, there is some $\eta \in \mathrm{Fm}^1_\Sigma$ such that $\eta^{\mathfrak{A}^2}(\langle 1, \frac{1}{2} \rangle) \in \{\langle \frac{1}{2}, 0 \rangle, \langle 0, 0 \rangle, \langle 0, \frac{1}{2} \rangle\}$, in which case $\sim^{\mathfrak{A}^2} \eta^{\mathfrak{A}^2}(\langle 1, \frac{1}{2} \rangle) = \langle 1, 1 \rangle$, and so $(I \times \{1\}) = \sim^{\mathfrak{B}} \eta^{\mathfrak{B}}((1 \wr \frac{1}{2})) \in B$, as required. □

Finally, consider the respective complementary cases:

(i) 2 forms a subalgebra of $\mathfrak{A}$.
    Consider the following complementary subcases:
    - $\sim^{\mathfrak{A}} \frac{1}{2} \neq \frac{1}{2}$.
      Then, by Lemma 6.1(ii) and Claim 6.3, $\mathcal{A} \restriction 2$ is embeddable into $\mathcal{B}$.
    - $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$,
      in which case $b \triangleq (1 \wr \frac{1}{2}) \in B \ni c \triangleq \sim^{\mathfrak{B}} b = (0 \wr \frac{1}{2})$. Consider the following complementary subsubcases:
      - $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$.
        Then, as $J \neq \varnothing$, $\{\langle e, (e \wr \frac{1}{2}) \rangle \mid e \in 2\}$ is an embedding of $\mathcal{A} \restriction 2$ into $\mathcal{B}$.
      - $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$.
        Then, there is some $\psi \in \mathrm{Fm}^1_\Sigma$ such that $\psi^{\mathfrak{A}}(\frac{1}{2}) \in 2$, in which

case $\psi^{\mathfrak{A}}(0) \in 2 \ni \psi^{\mathfrak{A}}(1)$, for 2 forms a subalgebra of $\mathfrak{A}$, and so, as $|2| = 2$, we have just the following exhaustive subsubsubcases:

* $\psi^{\mathfrak{A}}(\frac{1}{2}) = \psi^{\mathfrak{A}}(0)$,
  in which case, for some $x \in \{0,1\}$, $(I \times \{x\}) = (x \wr x) = \psi^{\mathfrak{B}}(c) \in B$, and so $\mathcal{A}{\upharpoonright}2$ is embeddable into $\mathcal{B}$, in view of Lemma 6.1(ii).
* $\psi^{\mathfrak{A}}(\frac{1}{2}) = \psi^{\mathfrak{A}}(1)$,
  in which case, for some $x \in \{0,1\}$, $(I \times \{x\}) = (x \wr x) = \psi^{\mathfrak{B}}(b) \in B$, and so $\mathcal{A}{\upharpoonright}2$ is embeddable into $\mathcal{B}$, in view of Lemma 6.1(ii).
* $\psi^{\mathfrak{A}}(1) = \psi^{\mathfrak{A}}(0)$,
  in which case, for some $x \in \{0,1\}$, $(I \times \{x\}) = (x \wr x) = \psi^{\mathfrak{B}}(\psi^{\mathfrak{B}}(c)) \in B$, and so $\mathcal{A}{\upharpoonright}2$ is embeddable into $\mathcal{B}$, in view of Lemma 6.1(ii).

(ii) 2 does not form a subalgebra of $\mathfrak{A}$.
Then, $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$, in view of Lemma 6.1(i) and Claim 6.3. Therefore, $b \triangleq (1 \wr \frac{1}{2}) \in B \ni c \triangleq \sim^{\mathfrak{B}} b = (0 \wr \frac{1}{2})$. And what is more, there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(0,1) = \frac{1}{2}$, in which case $\phi \triangleq \varphi(x_0, \sim x_0) \in \mathrm{Fm}^1_\Sigma$ and $\phi^{\mathfrak{A}}(0) = \frac{1}{2}$, and so $\phi^{\mathfrak{A}}(\frac{1}{2}) \neq \frac{1}{2}$, for, otherwise, we would have $B \ni \phi^{\mathfrak{B}}(c) = (\frac{1}{2} \wr \frac{1}{2})$, and so we would get $\sim^{\mathfrak{B}}(\frac{1}{2} \wr \frac{1}{2}) = (\frac{1}{2} \wr \frac{1}{2}) \in D^{\mathcal{B}}$, contrary to the non-$\sim$-paraconsistency and consistency of $\mathcal{B}$. In this way, $f \triangleq (\frac{1}{2} \wr 0) \in \{\phi^{\mathfrak{B}}(c), \sim^{\mathfrak{B}}\phi^{\mathfrak{B}}(c)\} \subseteq B$, in which case $g \triangleq \sim^{\mathfrak{B}} f = (\frac{1}{2} \wr 1) \in D^{\mathcal{B}}$, and so, by the non-$\sim$-paraconsistency and consistency of $\mathcal{B}$, we get $f = \sim^{\mathfrak{B}} g \notin D^{\mathcal{B}}$. Hence, $J \neq I$. Let us prove, by contradiction, that L_4 forms a subalgebra of $\mathfrak{A}^2$. For suppose L_4 does not form a subalgebra of $\mathfrak{A}^2$. Then, $\mathcal{B}$ is $\sim$-negative. Moreover, there is some $\xi \in \mathrm{Fm}^4_\Sigma$ such that $\xi^{\mathfrak{A}^2}(\langle\frac{1}{2},0\rangle, \langle\frac{1}{2},1\rangle, \langle 0,\frac{1}{2}\rangle, \langle 1,\frac{1}{2}\rangle) \in (A^2 \setminus L_4)$, in which case $B \ni b' \triangleq \xi^{\mathfrak{B}}(f,g,c,b) = (x \wr y)$, where $\langle x,y\rangle \in (A^2 \setminus L_4) = (2^2 \cup \{\frac{1}{2}\}^2)$, and so either $\sim^{\mathfrak{B}} b' = b' \in D^{\mathcal{B}}$, if $x = \frac{1}{2} = y$, or, otherwise, in which case $x, y \in \{0,1\}$, and so $x \neq y$, by Lemma 6.1(i), neither b' nor $\sim^{\mathfrak{B}} b' = (y \wr x)$ is in $D^{\mathcal{B}}$, for $J \neq \varnothing \neq (I \setminus J)$. This contradicts to the $\sim$-negativity of $\mathcal{B}$. Thus, L_4 forms a subalgebra of $\mathfrak{A}^2$. Hence, as $J \neq \varnothing \neq (I \setminus J)$, $\{\langle\langle w,z\rangle, (w \wr z)\rangle \mid \langle w,z\rangle \in L_4\}$ is an embedding of $\mathcal{A}^2{\upharpoonright}L_4$ into $\mathcal{B}$. □

Corollary 6.4. *Let I be a set, $\mathcal{B}$ a submatrix of $\mathcal{A}^I$, $\mathcal{D}$ a $\sim$-classical Σ-matrix and $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B},\mathcal{D})$. Suppose C is not $\sim$-classical. Then, the following hold:*

(i) *if 2 forms a subalgebra of $\mathfrak{A}$, then $\mathcal{A}{\upharpoonright}2$ is isomorphic to $\mathcal{D}$;*
(ii) *if 2 does not form a subalgebra of $\mathfrak{A}$, then $\mathcal{A}$ is false-singular, while L_4 forms a subalgebra of $\mathfrak{A}^2$, whereas $\theta^{\mathcal{A}^2{\upharpoonright}L_4} \in \mathrm{Con}(\mathfrak{A}^2{\upharpoonright}L_4)$, $(\mathcal{A}^2{\upharpoonright}L_4)/\theta^{\mathcal{A}^2{\upharpoonright}L_4}$ being isomorphic to $\mathcal{D}$.*

Proof. In that case, $\mathcal{B}$ is both $\sim$-negative, truth-non-empty and consistent, for $\mathcal{D}$ is so, and so is non-$\sim$-paraconsistent. And what is more, by (2.13), the logic C' of $\mathcal{B}$ is a $\sim$-classical extension of C, in which case C, being both non-$\sim$-classical and inferentially consistent, for $\mathcal{A}$ is both consistent and truth-non-empty, is not an extension of C', in view of Corollary 2.10, and so $\mathcal{A}$ is not a model of C'. Consider the respective complementary cases:

(i) 2 forms a subalgebra of $\mathfrak{A}$.
Then, by Lemmas 6.1 and 6.2(i), there is some $g \in \hom_{\mathrm{S}}(\mathcal{A}{\upharpoonright}2,\mathcal{B})$, in which case $(h \circ g) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}{\upharpoonright}2,\mathcal{D})$, for any $\sim$-classical Σ-matrix has no proper submatrix, and so Remark 2.5 completes the argument.

(ii) 2 does not form a subalgebra of $\mathfrak{A}$.
Then, by Lemma 6.1(i), $\mathcal{A}$ is false-singular, in which case, by Lemma 6.2(ii), L_4 forms a subalgebra of $\mathfrak{A}^2$, while there is an embedding e of $\mathcal{E} \triangleq (\mathcal{A}^2{\restriction}L_4)$ into $\mathcal{B}$, and so $g \triangleq (h \circ e) \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{E}, \mathcal{D})$, for any $\sim$-classical Σ-matrix has no proper submatrix, and so $(\ker g) \in \mathrm{Con}(\mathfrak{E})$. On the other hand, $(\ker g) = \theta \triangleq \theta^{\mathcal{E}}$, for $\mathcal{D}$ is both false- and truth-singular, so, by the Homomorphism Theorem, $g \circ \nu_\theta^{-1}$ is an isomorphism from $\mathcal{E}/\theta$ onto $\mathcal{D}$, as required. □

Theorem 6.5. *C is $\sim$-subclassical iff either of the following hold:*

(i) *C is $\sim$-classical;*
(ii) *2 forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{A}{\restriction}2$ is a $\sim$-classical model of C isomorphic to any that of C, and so defines a unique $\sim$-classical extension of C;*
(iii) *$\mathcal{A}$ is false-singular, while L_4 forms a subalgebra of $\mathfrak{A}^2$, whereas $\theta^{\mathcal{A}^2{\restriction}L_4} \in \mathrm{Con}(\mathfrak{A}^2{\restriction}L_4)$, in which case $(\mathcal{A}^2{\restriction}L_4)/\theta^{\mathcal{A}^2{\restriction}L_4}$ is a $\sim$-classical model of C isomorphic to any that of C, and so defines a unique $\sim$-classical extension of C.*

Proof. In case C is $\sim$-classical, the "'in which case" part of both (ii) and (iii) is by (2.13) and Lemma 2.9. In general, the "if" part is immediate.

Now, assume C is not $\sim$-classical. Consider any $\sim$-classical model $\mathcal{D}$ of C, in which case it is finite and simple. Hence, by Lemmas 2.7, 2.8 and Remark 2.5, there are some set I, some submatrix $\mathcal{B}$ of $\mathcal{A}^I$ and some $h \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{B}, \mathcal{D})$. Then, (2.13) and Corollary 6.4 complete the argument. □

In this way, by Theorem[s] 5.3 [and 6.5], we get effective algebraic criteria of C's being $\sim$-[sub]classical. On the other hand, the item (i) of Theorem 6.5 does not exhaust all $\sim$-subclassical three-valued (even $\sim$-paraconsistent) Σ-logics, as it ensues from:

Example 6.6. Let $i \in 2$, $\Sigma \triangleq \{\uplus, \sim\}$ with binary $\uplus$, $\mathcal{B}$ the $\sim$-classical Σ-matrix with $(j \uplus^{\mathfrak{B}} k) \triangleq i$, for all $j, k \in 2$, $D^{\mathcal{A}} \triangleq \{1, \frac{1}{2}\}$, $\sim^{\mathfrak{A}} \frac{1}{2} \triangleq \frac{1}{2}$ and

$$(a \uplus^{\mathfrak{A}} b) \triangleq \begin{cases} i & \text{if } a = \frac{1}{2}, \\ \frac{1}{2} & \text{otherwise,} \end{cases}$$

for all $a, b \in A$. Then, we have:

$$\begin{array}{rcl} (\langle \frac{1}{2}, a \rangle \uplus^{\mathfrak{A}^2} \langle b, \frac{1}{2} \rangle) & = & \langle i, \frac{1}{2} \rangle, \\ (\langle b, \frac{1}{2} \rangle \uplus^{\mathfrak{A}^2} \langle \frac{1}{2}, a \rangle) & = & \langle \frac{1}{2}, i \rangle, \\ (\langle \frac{1}{2}, a \rangle \uplus^{\mathfrak{A}^2} \langle \frac{1}{2}, b \rangle) & = & \langle i, \frac{1}{2} \rangle, \\ (\langle a, \frac{1}{2} \rangle \uplus^{\mathfrak{A}^2} \langle b, \frac{1}{2} \rangle) & = & \langle \frac{1}{2}, i \rangle, \end{array}$$

for all $a, b \in 2$. Therefore, L_4 forms a subalgebra of $\mathfrak{A}^2$ and $h \triangleq \chi^{\mathcal{A}^2{\restriction}L_4} \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{A}^2{\restriction}L_4, \mathcal{B})$, in which case $\theta^{\mathcal{A}^2{\restriction}L_4} = (\ker h) \in \mathrm{Con}(\mathfrak{A}^2{\restriction}L_4)$, and so C is $\sim$-subclassical, by Theorem 6.5. However, $(0 \uplus^{\mathfrak{A}} 1) = \frac{1}{2}$, so 2 does not form a subalgebra of $\mathfrak{A}$. □

Taking Lemma 2.9 and Theorem 6.5 into account, in case C is $\sim$-subclassical, the unique $\sim$-classical extension of C is denoted by $C^{\mathrm{PC}} = [\neq]C$, whenever C is [not] $\sim$-classical.

Corollary 6.7. *Suppose $\mathcal{A}$ is truth-singular. Then, the following are equivalent:*

(i) *C is inferentially maximal;*
(ii) *C is either $\sim$-classical or not $\sim$-subclassical;*

(iii) *either* 2 *does not form a subalgebra of* $\mathfrak{A}$ *or* C *is* $\sim$*-classical.*

In particular, C *is maximal iff both* C *has a theorem and either* 2 *does not form a subalgebra of* $\mathfrak{A}$ *or* C *is* $\sim$*-classical.*

Proof. First, (ii) is a particular case of (i). Next, (ii)$\Rightarrow$(iii) is by Theorem 6.5.

Finally, assume (iii) holds. Then, in case C is $\sim$-classical, (i) is by Corollary 2.10. Now, assume 2 does not form a subalgebra of $\mathfrak{A}$. Let C' be an inferentially consistent extension of C. Then, $x_1 \notin T \triangleq C'(x_0) \ni x_0$. On the other hand, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of C' (in particular, of C), and so is its finitely-generated consistent truth-non-empty submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^{2}_{\Sigma}, T \cap \mathrm{Fm}^{2}_{\Sigma}\rangle$, in view of (2.13). Hence, by Lemma 2.8, there is some set I and some submatrix $\mathcal{D} \in \mathbf{H}(\mathbf{H}^{-1}(\mathcal{B}))$ of $\mathcal{A}^I$, in which case $\mathcal{D}$ is a consistent truth-non-empty model of C', in view of (2.13), and so Lemma 6.1(i) and Remark 2.2(ii) complete the argument. $\Box$

In case $\mathcal{A}$ is truth-singular, this collectively with Theorem 5.3 provide effective algebraic criteria of the [inferential] maximality of C, because the set of all unary secondary operations of $\mathfrak{A}$ is finite. On the other hand, checking whether the image of one of them is equal to $\{1\}$ can be replaced by the much more simple procedure arising from the following particular case of Proposition 3.29 covering all three-valued $\veebar$-disjunctive $(\veebar, \sim)$-paracomplete Σ- logics with subclassical negation $\sim$:

Corollary 6.8. *Suppose* $\mathcal{A}$ *is both truth-singular and* $\veebar$*-disjunctive. Then,* C *is purely-inferential iff* $\{\frac{1}{2}\}$ *forms a subalgebra of* $\mathfrak{A}$*.*

7. Paraconsistent extensions

First, as $\mathcal{A}$ has no proper $\sim$-paraconsistent submatrix, by Theorems 3.26 and 4.1, we immediately have:

Corollary 7.1. *Any [non-]non-*$\sim$*-paraconsistent three-valued* Σ*-logic with subclassical negation* $\sim$ *has no* $\sim$*-paraconsistent [proper axiomatic] extension [and so is axiomatically maximally* $\sim$*-paraconsistent].*

Lemma 7.2. *Let* $\mathcal{B}$ *be a finitely-generated* $\sim$*-paraconsistent model of* C*. Suppose either* $\mathfrak{A}$ *has a ternary* $\frac{1}{2}$*-relative semi-conjunction or* $\{\frac{1}{2}\}$ *does not form a subalgebra of* $\mathfrak{A}$*. Then,* $\mathcal{A}$ *is embeddable into a strict surjective homomorphic image of* $\mathcal{B}$*.*

Proof. In that case, C is $\sim$-paraconsistent, in which case it is not $\sim$-classical, and so $\mathcal{A}$ is simple, by Theorem 5.3. Then, by Lemmas 2.7 and 2.8, there are some non-empty set I, some I-tuple $\overline{\mathcal{C}}$ constituted by submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$, some strict surjective homomorphic image $\mathcal{E}$ of $\mathcal{B}$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{E})$, in which case, by (2.13), $\mathcal{D}$ is $\sim$-paraconsistent, and so there are some $a \in D^{\mathcal{D}}$ such that $\sim^{\mathfrak{D}} a \in D^{\mathcal{D}}$ and some $b \in (D \setminus D^{\mathcal{D}})$. Then, $D \ni a = (I \times \{\frac{1}{2}\})$. Consider the following complementary cases:

- $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$,
  in which case $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$. Then, $\mathfrak{A}$ has a ternary $\frac{1}{2}$-relative semi-conjunction φ. Put $c \triangleq \varphi^{\mathfrak{D}}(b, \sim^{\mathfrak{D}} b, a) \in D$, $d \triangleq \sim^{\mathfrak{D}} c \in D$, $J \triangleq \{i \in I \mid \pi_i(b) = 1\}$ and $K \triangleq \{i \in I \mid \pi_i(b) = 0\} \neq \varnothing$, for $b \notin D^{\mathcal{D}}$. Given any $\bar{a} \in A^3$, set $(a_0 \wr a_1 \wr a_2) \triangleq ((J \times \{a_0\}) \cup (K \times \{a_1\}) \cup ((I \setminus (J \cup K)) \times \{a_2\})) \in A^I$. Then, $a = (\frac{1}{2} \wr \frac{1}{2} \wr \frac{1}{2})$ and $b = (1 \wr 0 \wr \frac{1}{2})$. Consider the following exhaustive subcases:
  - $\varphi^{\mathfrak{A}}(1, 0, \frac{1}{2}) = 0$,
    in which case we have $c = (0 \wr 0 \wr \frac{1}{2})$ and $d = (1 \wr 1 \wr \frac{1}{2})$, and so, since

$K \neq \varnothing$, while $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$, $f \triangleq \{\langle e, (e \wr e \wr \frac{1}{2})\rangle \mid e \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$.

– $\varphi^{\mathfrak{A}}(1, 0, \frac{1}{2}) = \frac{1}{2}$,
in which case we have $c = (\frac{1}{2} \wr 0 \wr \frac{1}{2})$ and $d = (\frac{1}{2} \wr 1 \wr \frac{1}{2})$, and so, since $K \neq \varnothing$, while $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$, $f \triangleq \{\langle e, (\frac{1}{2} \wr e \wr \frac{1}{2})\rangle \mid e \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$.

- $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$.
Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\frac{1}{2}) \neq \frac{1}{2}$, in which case $\{\frac{1}{2}, \varphi^{\mathfrak{A}}(\frac{1}{2}), {\sim}^{\mathfrak{A}}\varphi^{\mathfrak{A}}(\frac{1}{2})\} = A$, and so $D \supseteq \{a, \varphi^{\mathfrak{D}}(a), {\sim}^{\mathfrak{D}}\varphi^{\mathfrak{D}}(a)\} = \{I \times \{e\} \mid e \in A\}$. Therefore, as $I \neq \varnothing$, $f \triangleq \{\langle e, I \times \{e\}\rangle \mid e \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$.

Then, $(g \circ f) \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{E})$ is injective, by Remark 2.5. □

Theorem 7.3. *Suppose $\mathcal{A}$ is false-singular (in particular, $\sim$-paraconsistent) [and C is $\sim$-subclassical]. Then, the following are equivalent:*

(i) *C has no proper $\sim$-paraconsistent [$\sim$-subclassical] extension;*
(ii) *C has no proper $\sim$-paraconsistent non-$\sim$-subclassical extension;*
(iii) *either $\mathcal{A}$ has a ternary $\frac{1}{2}$-relative semi-conjunction or $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$ (in particular, ${\sim}^{\mathfrak{A}}\frac{1}{2} \neq \frac{1}{2}$);*
(iv) *$L_3 \triangleq \{\langle\frac{1}{2}, \frac{1}{2}\rangle, \langle 0, 1\rangle, \langle 1, 0\rangle\}$ does not form a subalgebra of $\mathfrak{A}^2$;*
(v) *$\mathcal{A}$ has no truth-singular $\sim$-paraconsistent subdirect square;*
(vi) *$\mathcal{A}^2$ has no truth-singular $\sim$-paraconsistent submatrix;*
(vii) *C has no truth-singular $\sim$-paraconsistent model.*

In particular, C has a $\sim$-paraconsistent proper extension iff it has a [non-]non-$\sim$-subclassical one.

Proof. First, assume (iii) holds. Consider any $\sim$-paraconsistent extension C' of C, in which case $x_1 \notin T \triangleq C'(\{x_0, {\sim}x_0\}) \supseteq \{x_0, {\sim}x_0\}$, while, by the structurality of C', $\langle \mathfrak{Fm}^\omega_\Sigma, T\rangle$ is a model of C' (in particular, of C), and so is its finitely-generated $\sim$-paraconsistent submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^2_\Sigma, T \cap \mathrm{Fm}^2_\Sigma\rangle$, in view of (2.13). Then, by Lemma 7.2 and (2.13), $\mathcal{A}$ is a model of C', and so $C' = C$. Thus, both (i) and (ii) hold.

Next, assume L_3 forms a subalgebra of $\mathfrak{A}^2$. Then, $\mathcal{B} \triangleq (\mathcal{A}^2 \upharpoonright L_3)$ is a subdirect square of $\mathcal{A}$. Moreover, as $L_3 \ni \langle 0, 1\rangle \notin (L_3 \cap \Delta_A) = \{\langle\frac{1}{2}, \frac{1}{2}\rangle\} = D^{\mathcal{B}}$, for $\mathcal{A}$ is false-singular, $\mathcal{B}$ is both truth-singular and $\sim$-paraconsistent. Thus, (v)⇒(iv) holds, while (v) is a particular case of (vi), whereas (vii)⇒(vi) is by (2.13).

Now, let $\mathcal{B} \in \mathrm{Mod}(C)$ be both $\sim$-paraconsistent and truth-singular, in which case the rule $x_0 \vdash {\sim}x_0$ is true in $\mathcal{B}$, and so is its logical consequence $\{x_0, x_1, {\sim}x_1\} \vdash {\sim}x_0$, not being true in $\mathcal{A}$ under $[x_0/1, x_1/\frac{1}{2}]$ [but true in any $\sim$-classical model $\mathcal{C}'$ of C, for $\mathcal{C}'$ is $\sim$-negative]. Thus, the logic of $\{\mathcal{B}[, \mathcal{C}']\}$ is a proper $\sim$-paraconsistent [$\sim$-subclassical] extension of C, so (i)⇒(vii) holds. And what is more, $x_0 \vdash {\sim}x_0$, being true in $\mathcal{B}$, is true in nether $\mathcal{A}$ under $[x_0/1]$ nor any $\sim$-classical Σ-matrix $\mathcal{C}''$ under $[x_0/1_{\mathcal{C}''}]$. Thus, the logic of $\mathcal{B}$ is a proper $\sim$-paraconsistent non-$\sim$-subclassical extension of C, so (ii)⇒(vii) holds.

Finally, assume $\mathcal{A}$ has no ternary $\frac{1}{2}$-relative semi-conjunction and $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$. In that case, ${\sim}^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}^2$ generated by L_3. If $\langle 0, 0\rangle$ was in B, then there would be some $\varphi \in \mathrm{Fm}^3_\Sigma$ such that $\varphi^{\mathfrak{A}}(0, 1, \frac{1}{2}) = 0 = \varphi^{\mathfrak{A}}(1, 0, \frac{1}{2})$, in which case it would be a ternary $\frac{1}{2}$-relative semi-conjunction for $\mathcal{A}$. Likewise, if either $\langle\frac{1}{2}, 0\rangle$ or $\langle 0, \frac{1}{2}\rangle$ was in B, then there would be some $\varphi \in \mathrm{Fm}^3_\Sigma$ such that $\varphi^{\mathfrak{A}}(0, 1, \frac{1}{2}) = 0$ and $\varphi^{\mathfrak{A}}(1, 0, \frac{1}{2}) = \frac{1}{2}$, in which case it would be a ternary $\frac{1}{2}$-relative semi-conjunction for $\mathcal{A}$. Therefore, as ${\sim}^{\mathfrak{A}}1 = 0$ and ${\sim}^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$, we conclude that $(\{\langle 0,\frac{1}{2}\rangle,\langle 1,\frac{1}{2}\rangle,\langle\frac{1}{2},1\rangle,\langle\frac{1}{2},0\rangle,\langle 0,0\rangle,\langle 1,1\rangle\}\cap B)=\varnothing$. Thus, $B=L_3$ forms a subalgebra of $\mathfrak{A}^2$. In this way, (iv)⇒(iii) holds. □

Theorem 7.3(i)⇔(iii[iv]) is especially useful for [effective dis]proving the maximal $\sim$-paraconsistency of C [cf. Example 10.10].

8. Non-subclassical consistent extensions

In case C is not $\sim$-subclassical, it, being [inferentially] consistent, for $\mathcal{A}$ is [both] so [and truth-non-empty], is clearly a[n inferentially] consistent non-$\sim$-subclassical extension of itself. Here, we explore the opposite case.

Theorem 8.1. *Let C' be an inferentially consistent extension of C. Suppose $\mathcal{A}$ is truth-singular and C is $\sim$-subclassical. Then, C' is a sublogic of C^{PC}.*

Proof. The case, when $C'=C$, is by the inclusion $C\subseteq C^{\mathrm{PC}}$. Now, assume $C'\neq C$. Then, $x_1\notin T\triangleq C'(x_0)\ni x_0$. On the other hand, by the structurality of C', $\langle\mathfrak{Fm}^{\omega}_{\Sigma},T\rangle$ is a model of C' (in particular, of C), and so is its finitely-generated consistent truth-non-empty submatrix $\mathcal{B}\triangleq\langle\mathfrak{Fm}^2_{\Sigma},T\cap\mathrm{Fm}^2_{\Sigma}\rangle$, in view of (2.13). Hence, by Lemma 2.8, there is some set I and some submatrix $\mathcal{D}\in\mathbf{H}(\mathbf{H}^{-1}(\mathcal{B}))$ of $\mathcal{A}^I$, in which case $\mathcal{D}$ is a consistent truth-non-empty model of C', in view of (2.13), and so $\mathcal{A}$ is not a model of the logic of $\mathcal{D}$, for $C'\neq C$. In this way, (2.13), Lemma 6.1 and Theorem 6.5 complete the argument. □

Since C is inferentially consistent, for $\mathcal{A}$ is both consistent and truth-non-empty, by Remark 2.2(ii) and Theorem 8.1, we immediately get:

Corollary 8.2. *Suppose $\mathcal{A}$ is truth-singular and C is $\sim$-subclassical. Then, C has a consistent non-$\sim$-subclassical (viz, not being a sublogic of C^{PC}; cf. Lemma 2.9 and Theorem 6.5) extension iff C has no theorem.*

In case $\mathcal{A}$ is truth-singular [and $\veebar$-disjunctive], this provides a [quite] effective criterion of C's having a consistent non-$\sim$-subclassical extension [cf. Corollary 6.8]. On the other hand, as we show below, in case $\mathcal{A}$ is false-singular, such a criterion holds as well, but becoming quite effective, even if $\mathcal{A}$ is not $\veebar$-disjunctive.

Lemma 8.3. *Let $\mathcal{B}$ be a $\sim$-classical Σ-matrix and C' the logic of $\mathcal{B}$. Then, the following are equivalent:*

(i) *C' has a theorem;*
(ii) *there is some $\phi\in\mathrm{Fm}^2_{\Sigma}$ such that $\phi(x_0,\sim x_0)$ is a theorem of C';*
(iii) *$B^2\setminus\Delta_B$ does not form a subalgebra of $\mathfrak{B}^2$;*
(iv) *$\mathcal{B}$ has no truth-empty model.*

Proof. First, (i) is a particular case of (ii). Next, (i)⇒(iv) is by Remark 2.4.

Further, in case $D\triangleq\{\langle 0,1\rangle,\langle 1,0\rangle\}=(B^2\setminus\Delta_B)\subseteq(B^2\setminus\{\langle 1,1\rangle\})=(B^2\setminus D^{\mathcal{B}^2})$ forms a subalgebra of $\mathfrak{B}^2$, by (2.13), $\mathcal{D}\triangleq(\mathcal{B}^2{\restriction}D)$ is a truth-empty model of C'. Thus, (iv)⇒(iii) holds.

Finally, assume (iii) holds, in which case there is some $\psi\in\mathrm{Fm}^2_{\Sigma}$ such that $\psi^{\mathfrak{B}}(0,1)=(0|1)=\psi^{\mathfrak{B}}(1,0)$, and so, respectively, $\phi\triangleq\sim^{1|0}\psi\in\mathrm{Fm}^2_{\Sigma}$, while $\phi(x_0,\sim x_0)$ is a theorem of C'. Thus, (ii) holds, as required. □

To unify further notations, set $L_2\triangleq 2$.

Theorem 8.4. *Suppose $\mathcal{A}$ is false-singular, while C is both $\sim$-subclassical and non-$\sim$-classical (in which case $L_{2[+2]}$ forms a subalgebra of $\mathfrak{A}^{[2]}$; cf. Theorem 6.5). Then, the following are equivalent:*

(i) *C has a consistent non-$\sim$-subclassical (viz, not being a sublogic of C^{PC}; cf. Theorem 6.5) extension;*

(ii) $\mathfrak{A}$ *has no binary semi-conjunction (in particular,* C *has a proper* $\sim$*-paraconsistent* $\{\sim$*-subclassical*$\}$ *extension; cf. Theorem 7.3);*

(iii) $M_2 \triangleq \{\langle 0,1\rangle,\langle 1,0\rangle\}$ *[resp.,* $M_8 \triangleq \{\langle\{\langle i,\frac{1}{2}\rangle,\langle 1-i,j\rangle\},\{\langle k,\frac{1}{2}\rangle,$ $\langle 1-k,1-j\rangle\}\rangle \mid i,j,k \in 2\}$*] forms a subalgebra of* $(\mathfrak{A}^{[2]}{\restriction}L_{2[+2]})^2$*;*

(iv) C^{PC} *has a truth-empty model;*

(v) C^{PC} *has no theorem;*

(vi) C *has a truth-empty model;*

(vii) C *has no theorem.*

In particular, C *has a truth-empty model/theorem iff* C^{PC} *does so/ iff* C *has no truth-empty model.*

Proof. First, assume $\mathfrak{A}$ has a binary semi-conjunction. Consider any consistent extension C' of C. In case $C' = C$, we have $C' = C \subseteq C^{\mathrm{PC}}$. Now, assume $C' \neq C$, in which case C' is non-$\sim$-paraconsistent, by Theorem 7.3. Then, as C' is consistent, we have $x_0 \notin C'(\varnothing)$, while, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, C'(\varnothing)\rangle$ is a model of C' (in particular, of C), and so is its consistent finitely-generated submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^1_{\Sigma}, \mathrm{Fm}^1_{\Sigma} \cap C'(\varnothing)\rangle$, in view of (2.13). Hence, by Lemma 2.8, there are some set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D}$ of it such that $\mathcal{B}$ is a strict surjective homomorphic image of a strict surjective homomorphic counter-image of $\mathcal{D}$, in which case $\mathcal{D}$ is a consistent model of C', in view of (2.13), and so, a non-$\sim$-paraconsistent submatrix of $\mathcal{A}^I$. In particular, as $C' \neq C$, $\mathcal{A}$ is not a model of the logic of $\mathcal{D}$. Then, by (2.13), Lemma 6.2 and Theorem 6.5, a Σ-matrix defining C^{PC} is embeddable into $\mathcal{D}$, in which case $C' \subseteq C^{\mathrm{PC}}$, and so (i)$\Rightarrow$(ii) holds.

Next, assume C^{PC} has a theorem. Then, by Lemma 8.3(i)$\Rightarrow$(ii), there is some $\phi \in \mathrm{Fm}^2_{\Sigma}$ such that $\psi \triangleq \phi(x_0,\sim x_0)$ is a theorem of C^{PC}. Consider the following complementary cases:

- 2 forms a subalgebra of $\mathfrak{A}$,
  in which case, by Theorem 6.5(i), C^{PC} is defined by $\mathcal{A}{\restriction}2$, and so $\sim\phi$ is a binary semi-conjunction for $\mathfrak{A}$.
- 2 does not form a subalgebra of $\mathfrak{A}$,
  in which case, by (2.13) and Theorem 6.5, L_4 forms a subalgebra of $\mathfrak{A}^2$, while C^{PC} is defined by $\mathcal{B} \triangleq (\mathcal{A}^2{\restriction}L_4)$, and so $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$, in view of Claim 6.3, while, as $\langle\frac{1}{2},0/1\rangle \in L_4$, $a \triangleq \phi^{\mathfrak{A}^2}(\langle\frac{1}{2},0/1\rangle,\langle\frac{1}{2},1/0\rangle) = \psi^{\mathfrak{A}^2}(\langle\frac{1}{2},0/1\rangle) \in D^{\mathcal{B}} = \{\langle\frac{1}{2},1\rangle,\langle 1,\frac{1}{2}\rangle\}$. Consider the following complementary subcases:
  - $\psi^{\mathfrak{A}}(\frac{1}{2}) = \frac{1}{2}$,
    in which case $\psi^{\mathfrak{A}}(0/1) = 1$, and so $\sim\phi$ is a binary semi-conjunction for $\mathfrak{A}$.
  - $\psi^{\mathfrak{A}}(\frac{1}{2}) \neq \frac{1}{2}$,
    in which case $\psi^{\mathfrak{A}}(\frac{1}{2}) = 1$, while $\psi^{\mathfrak{A}}(0/1) = \frac{1}{2}$, and so $\sim\psi(\phi)$ is a binary semi-conjunction for $\mathfrak{A}$.

Thus, anyway, (ii) does not hold, and so (ii)$\Rightarrow$(v) holds.

Further, (iii)$\Leftrightarrow$(iv)$\Leftrightarrow$(v) are by Lemma 8.3(i)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv) and Theorem 6.5, while (iv)$\Rightarrow$(vi) is by the inclusion $C \subseteq C^{\mathrm{PC}}$, whereas (vi)$\Rightarrow$(vii) is by Remark 2.4.

Finally, (vii)$\Rightarrow$(i) is by Remark 2.2(ii) and the fact that C is inferentially consistent, for $\mathcal{A}$ is both consistent and truth-non-empty. □

Then, combining Corollary 8.2 and Theorem 8.4, we eventually get:

Corollary 8.5. *Suppose* C *is [not] non-*$\sim$*-subclassical. Then,* C *has a consistent non-*$\sim$*-subclassical [viz, not being a sublogic of* C^{PC}*; cf. Lemma 2.9 and Theorem 6.5] extension [iff* C *has no theorem].*

Theorem 8.6. *Suppose $\mathcal{A}$ is false-singular and C is both $\sim$-subclassical and non-$\sim$-classical. Then, any inferentially consistent extension of C is a sublogic of $C^{\rm PC}$ iff both $\mathcal{A}$ satisfies GC and L_3 does not form a subalgebra of $\mathfrak{A}^2$.*

Proof. First, assume $\mathcal{A}$ does not satisfy GC. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}^2$ generated by $\{\langle 1,\frac{1}{2}\rangle\}$, in which case $\mathcal{B} \triangleq (\mathcal{A}^2{\restriction}B)$ is a model of C, in view of (2.13). Moreover, $\langle 1,\frac{1}{2}\rangle \in D^{\mathcal{B}}$, for $\mathcal{A}$ is false-singular, in which case case $\mathcal{B}$ is truth-non-empty, while $\langle 0,\sim^{\mathfrak{A}}\frac{1}{2}\rangle = \sim^{\mathfrak{A}^2}\langle 1,\frac{1}{2}\rangle \in (B\setminus D^{\mathcal{B}})$, for $0 \notin D^{\mathcal{A}}$, and so $\mathcal{B}$ is consistent. And what is more, $D \triangleq (B\setminus D^{\mathcal{B}}) \subseteq \{\langle 0,1\rangle,\langle 1,0\rangle\}$, in which case, for each $b \in D$, $\sim^{\mathfrak{B}} b \in D$, and so the rule $\sim x_0 \vdash x_0$ is true in $\mathcal{B}$. On the other hand, this rule is not true in any $\sim$-classical Σ-matrix $\mathcal{C}'$ under $[x_0/0_{\mathcal{C}'}]$. Thus, the logic of $\mathcal{B}$ is an inferentially consistent non-$\sim$-subclassical extension of C.

Likewise, by Theorem 7.3, in case L_3 forms a subalgebra of $\mathfrak{A}^2$, C has a $\sim$-paraconsistent (in particular, inferentially consistent) non-$\sim$-subclassical extension.

Conversely, assume both $\mathcal{A}$ satisfies GC and L_3 does not form a subalgebra of $\mathfrak{A}^2$. Consider any inferentially consistent extension C' of C. In case $C' = C$, we have $C' = C \subseteq C^{\rm PC}$. Now, assume $C' \neq C$, in which case C' is non-$\sim$-paraconsistent, by Theorem 7.3. Then, as C' is inferentially consistent, we have $x_1 \notin C'(x_0) \ni x_0$, while, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, C'(x_0)\rangle$ is a model of C' (in particular, of C), and so is its consistent truth-non-empty finitely-generated submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^2_{\Sigma}, {\rm Fm}^2_{\Sigma} \cap C'(x_0)\rangle$, in view of (2.13). Hence, by Lemma 2.8, there are some set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D}$ of it such that $\mathcal{B}$ is a strict surjective homomorphic image of a strict surjective homomorphic counter-image of $\mathcal{D}$, in which case $\mathcal{D}$ is a consistent truth-non-empty model of C', in view of (2.13), and so, a non-$\sim$-paraconsistent submatrix of $\mathcal{A}^I$. In particular, as $C' \neq C$, $\mathcal{A}$ is not a model of the logic of $\mathcal{D}$. Then, by (2.13), Lemma 6.2 and Theorem 6.5, a Σ-matrix defining $C^{\rm PC}$ is embeddable into $\mathcal{D}$, in which case $C' \subseteq C^{\rm PC}$. □

In this way, summing up Theorems 8.1, 8.6 and Corollary 2.10, we eventually get the following "inferential" analogue of Corollary 8.5:

Corollary 8.7. *Suppose C is [not] non-$\sim$-subclassical. Then, C has an inferentially consistent non-$\sim$-subclassical [viz, not being a sublogic of $C^{\rm PC}$; cf. Lemma 2.9 and Theorem 6.5] extension [iff neither C is $\sim$-classical nor $\mathcal{A}$ is truth-singular nor both $\mathcal{A}$ satisfies GC and L_3 does not form a subalgebra of $\mathfrak{A}^2$].*

9. Conjunctive three-valued logics with subclassical negation

Remark 9.1. If $\mathcal{A}$ is weakly $\bar{\wedge}$-conjunctive and false-singular, then we have $(0 \bar{\wedge}^{\mathfrak{A}} \frac{1}{2}) = 0 = (\frac{1}{2} \bar{\wedge}^{\mathfrak{A}} 0)$, in which case we get $(\langle 0,\frac{1}{2}\rangle \bar{\wedge}^{\mathfrak{A}^2} \langle \frac{1}{2},0\rangle) = \langle 0,0\rangle \notin L_4 \supseteq \{\langle 0,\frac{1}{2}\rangle,\langle\frac{1}{2},0\rangle\}$, and so L_4 does not form a subalgebra of $\mathfrak{A}^2$. □

By Theorem 6.5 and Remark 9.1, we immediately have:

Corollary 9.2. *Suppose C is weakly $\bar{\wedge}$-conjunctive (viz., $\mathcal{A}$ is so) and not $\wr$-classical. Then, C is $\sim$-subclassical iff 2 forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{A}{\restriction}2$ is isomorphic to any $\sim$-classical model of C, and so defines a unique $\sim$-classical extension of C, that is, $C^{\rm PC}$.*

Likewise, by (2.13), Lemma 2.9, Theorems 5.3, 6.5 and Remark 9.1, we also have:

Corollary 9.3. *Suppose $\mathcal{A}$ is weakly $\bar{\wedge}$-conjunctive (viz., C is so) and false-singular. Then, C is $\sim$-subclassical iff either of the following hold:*

(i) $\theta^{\mathcal{A}} \in {\rm Con}(\mathfrak{A})$, *in which case $\mathcal{A}/\theta^{\mathcal{A}}$ is isomorphic to any $\sim$-classical model of C, and so defines a unique $\sim$-classical extension of C, that is, $C^{\rm PC}$;*

(ii) 2 *forms a subalgebra of* $\mathfrak{A}$*, in which case* $\mathcal{A}{\restriction}2$ *is isomorphic to any* $\sim$*-classical model of* C*, and so defines a unique* $\sim$*-classical extension of* C*, that is,* C^{PC}*.*

Remark 9.4. Suppose either $\mathcal{A}$ is both false-singular and weakly $\overline{\wedge}$-conjunctive or both 2 forms a subalgebra of $\mathfrak{A}$ and $\mathcal{A}{\restriction}2$ is weakly $\overline{\wedge}$-conjunctive. Then, $(x_0 \mathbin{\overline{\wedge}} x_1)$ is a binary semi-conjunction for $\mathcal{A}$. □

First, by Theorem 8.4 and Remark 9.4, we immediately have:

Corollary 9.5. *Let* C' *be a{n inferentially} consistent extension of* C*. Suppose* 2 *forms a subalgebra of* $\mathfrak{A}$ *(in which case* C *is* $\sim$*-subclassical; cf. Theorem 6.5),* $\mathcal{A}$ *is false-singular and* $\mathcal{A}{\restriction}2$ *is weakly* $\overline{\wedge}$*-conjunctive (in particular,* $\mathcal{A}$ *[viz.,* C*] is so; cf. Remark 2.6(ii)). Then,* C *has a/no theorem/truth-empty model, while* C^{PC} *is an extension of* C'*.*

Finally, by Theorems 4.1, 7.3 and Remark 9.4, we immediately get the following *universal* result, *properly* subsuming the reference [Pyn 95b] of [11]:

Corollary 9.6. *Any* $\sim$*-paraconsistent three-valued weakly* $\overline{\wedge}$*-conjunctive* Σ*-logic with subclassical negation* $\sim$ *is maximally so.*

The principal advance of the present study with regard to the reference [Pyn 95b] of [11] consists in proving inheritance of the maximal paraconsistency by *three-valued* expansions of [weakly] conjunctive paraconsistent three-valued logics with subclassical negation, because both paraconsistency, subclassical negation and [weak] conjunction are inherited by expansions, while the property of being subclassical is not, generally speaking, so. In particular, Corollary 9.6 implies the maximal paraconsistency of arbitrary three-valued expansions (cf. Corollary 5.9 in this connection) of LP, HZ and P^1 equally covered by this section, in general.

10. Disjunctive three-valued logics with subclassical negation

Lemma 10.1. *Let* $\mathcal{B}$ *be a* Σ*-matrix and* C' *the logic of* $\mathcal{B}$*. Suppose [either]* $\mathcal{B}$ *is false-singular (in particular,* $\sim$*-classical) [or both* $\mathcal{B}$ *is* $\sim$*-super-classical and* $|B| \leqslant 3$*]. Then, the following are equivalent:*

(i) C' *is* $\veebar$*-disjunctive;*
(ii) $\mathcal{B}$ *is* $\veebar$*-disjunctive;*
(iii) (2.3)*,* (2.4) *and* (2.5) *[as well as the* Resolution rule*:*

$$\{x_0 \veebar x_1, {\sim}x_0 \veebar x_1\} \vdash x_1] \tag{10.1}$$

are satisfied in C' *(viz., true in* $\mathcal{B}$*).*

Proof. First, (ii)⇒(i) is immediate.

Next, assume (i) holds. Then, (2.3), (2.4) and (2.5) are immediate. [In addition, suppose $\mathcal{B}$ is not false-singular, in which case it is $\sim$-super-classical, while $|B| \leqslant 3$, and so it is both truth-singular and, therefore, not $\sim$-paraconsistent. Hence, $x_1 \in (C'(x_1) \cap C'(\{x_0, {\sim}x_0\})) = (C'(x_1) \cap C'(\{x_0 \veebar x_1, {\sim}x_0\})) = C'(\{x_0 \veebar x_1, {\sim}x_0 \veebar x_1\})$, that is, (10.1) is satisfied in C'.] Thus, (iii) holds.

Finally, assume (iii) holds. Consider any $a, b \in B$. In case $(a/b) \in D^{\mathcal{B}}$, by (2.3)/ and (2.4), we have $(a \veebar^{\mathfrak{B}} b) \in D^{\mathcal{B}}$. Now, assume $(\{a, b\} \cap D^{\mathcal{B}}) = \varnothing$. Then, in case $a = b$ (in particular, $\mathcal{B}$ is false-singular), by (2.5), we get $D^{\mathcal{B}} \not\ni (a \veebar^{\mathfrak{B}} a) = (a \veebar^{\mathfrak{B}} b)$. [Otherwise, $\mathcal{B}$ is not false-singular, in which case it is $\sim$-super-classical, while $|B| \leqslant 3$, whereas (10.1) is true in $\mathcal{B}$, and so, for some $c \in (B \setminus D^{\mathcal{B}}) = \{a, b\}$, it holds that ${\sim}^{\mathfrak{B}} c \in D^{\mathcal{B}}$. Let d be the unique element of $\{a, b\} \setminus \{c\}$, in which case $\{a, b\} = \{c, d\}$. Then, since, by (2.3), we have $({\sim}^{\mathfrak{B}} c \veebar^{\mathfrak{B}} d) \in D^{\mathcal{B}}$, we conclude that $(c \veebar^{\mathfrak{B}} d) \notin D^{\mathcal{B}}$, for, otherwise, by (10.1), we would get $d \in D^{\mathcal{B}}$. Hence, by (2.4), we eventually get $(a \veebar^{\mathfrak{B}} b) \notin D^{\mathcal{B}}$.] Thus, (ii) holds, as required. □

Corollary 10.2. *Suppose C is $\veebar$-disjunctive (viz., $\mathcal{A}$ is so; cf. Lemma 10.1). Then, C is $\sim$-classical iff $\theta^{\mathcal{A}} \in \mathrm{Con}(\mathfrak{A})$, in which case $\mathcal{A}/\theta^{\mathcal{A}}$ is isomorphic to any $\sim$-classical model of C, and so defines a unique $\sim$-classical extension of C, that is, $C^{\mathrm{PC}} = C$.*

Proof. The "in which case" part is by (2.13) and Lemma 2.9. The "if" part is by Theorem 5.3. The converse is proved by contradiction. For suppose C is $\sim$-classical, while $\theta^{\mathcal{A}} \notin \mathrm{Con}(\mathfrak{A})$. Then, by Theorem 5.3, 2 forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{B} \triangleq (\mathcal{A}{\restriction}2)$ is $\veebar$-disjunctive, for $\mathcal{A}$ is so, and so $(0 \veebar^{\mathfrak{A}} 1) = 1 = (1 \veebar^{\mathfrak{A}} 0)$, while $\mathcal{B}^2$ is a strict surjective homomorphic counter-image of $\mathcal{A}$, in which case it is $\veebar$-disjunctive, for $\mathcal{A}$ is so, and so, as $(\{\langle 0,1\rangle, \langle 1,0\rangle\} \cap D^{\mathcal{B}^2}) = \varnothing$, we have $D^{\mathcal{B}^2} \not\ni (\langle 0,1\rangle \veebar^{\mathfrak{B}^2} \langle 1,0\rangle) = \langle 1,1\rangle \in D^{\mathcal{B}^2}$, as required. □

10.1. **Implicative three-valued logics with subclassical negation.**

Lemma 10.3. *Let $\mathcal{B}$ be a Σ-matrix and C' the logic of $\mathcal{B}$. Suppose [either] $\mathcal{B}$ is false-singular (in particular, $\sim$-classical) [or both $\mathcal{B}$ is $\sim$-super-classical and $|B| \leqslant 3$]. Then, the following [but (i)] are equivalent:*

(i) *C' is weakly $\sqsupset$-implicative;*
(ii) *C' is $\sqsupset$-implicative;*
(iii) *$\mathcal{B}$ is $\sqsupset$-implicative;*
(iv) (2.6), (2.7) *and* (2.8) *[as well as both* (2.9) *and the* Ex Contradictione Quodlibet *axiom:*

$$\sim x_0 \sqsupset (x_0 \sqsupset x_1)] \tag{10.2}$$

are satisfied in C' (viz., true in $\mathcal{B}$).

Proof. First, (iii)⇒(ii) is immediate, while (i) is a particular case of (ii).

Next, assume (i[i]) holds. Then, (2.6), (2.7) and (2.8) [as well as (2.9)] are immediate. [In addition, suppose $\mathcal{B}$ is not false-singular, in which case it is $\sim$-super-classical, while $|B| \leqslant 3$, and so it is both truth-singular and, therefore, non-$\sim$-paraconsistent, and so is C'. Hence, by Deduction Theorem, (10.2) is satisfied in C'.] Thus, (iv) holds.

Finally, assume (iv) holds. Consider any $a, b \in B$. In case $b \in D^{\mathcal{B}}$, by (2.7) and (2.8), we have $(a \sqsupset^{\mathfrak{B}} b) \in D^{\mathcal{B}}$. Likewise, in case $a \in D^{\mathcal{B}} \ni (a \sqsupset^{\mathfrak{B}} b)$, by (2.8), we have $b \in D^{\mathcal{B}}$. Now, assume $(\{a,b\} \cap D^{\mathcal{B}}) = \varnothing$. Then, in case $a = b$ (in particular, $\mathcal{B}$ is false-singular), by (2.6), we get $D^{\mathcal{B}} \not\ni (a \sqsupset^{\mathfrak{B}} a) = (a \sqsupset^{\mathfrak{B}} b)$. [Otherwise, $\mathcal{B}$ is not false-singular, in which case it is $\sim$-super-classical, while $|B| \leqslant 3$, whereas both (2.9) and (10.2) and true in $\mathcal{B}$, and so, for some $c \in (B \setminus D^{\mathcal{B}}) = \{a, b\}$, it holds that $\sim^{\mathfrak{B}} c \in D^{\mathcal{B}}$. Let d be the unique element of $\{a,b\} \setminus \{c\}$, in which case $\{a,b\} = \{c,d\}$. Then, since $\sim^{\mathfrak{B}} c \in D^{\mathcal{B}}$, by (10.2), we conclude that $(c \sqsupset^{\mathfrak{B}} d) \in D^{\mathcal{B}}$. Let us prove, by contradiction, that $(d \sqsupset^{\mathfrak{B}} c) \in D^{\mathcal{B}}$. For suppose $(d \sqsupset^{\mathfrak{B}} c) \notin D^{\mathcal{B}}$, in which case $(d \sqsupset^{\mathfrak{B}} c) = (c/d)$, and so we have $((d \sqsupset^{\mathfrak{B}} c) \sqsupset^{\mathfrak{B}} d) = ((c \sqsupset^{\mathfrak{B}} d)/(d \sqsupset^{\mathfrak{B}} d)) \in D^{\mathcal{B}}/$, by (2.6). Hence, by (2.8) and (2.9), we get $d \in D^{\mathcal{B}}$. This contradiction shows that $(d \sqsupset^{\mathfrak{B}} c) \in D^{\mathcal{B}} \ni (c \sqsupset^{\mathfrak{B}} d)$. In particular, we eventually get $(a \sqsupset^{\mathfrak{B}} b) \in D^{\mathcal{B}}$.] Thus, (iii) holds, as required. □

10.2. **Disjunctive versus classical extensions.** By C^{R} we denote the extension of C relatively axiomatized by (10.1).

Remark 10.4. Given any $\veebar$-disjunctive Σ-logic C', by (2.5)|both (2.3) and (2.4), applying $[x_1/x_0, x_2/x_1, x_0/x_1]|[x_1/x_0, x_0/x_1]$ to $(\sigma_{+1}(2.10) \veebar x_0)|(10.1)$, any extension of C' satisfies $(10.1)|(\sigma_{+1}(2.10) \veebar x_0)$, whenever it satisfies $(\sigma_{+1}(2.10) \veebar x_0)|(10.1)$. Hence, C^{R} is the extension of C relatively axiomatized by $\sigma_{+1}(2.10) \veebar x_0$. □

Theorem 10.5. *Let C' be an extension of C. Suppose C is $\veebar$-disjunctive (i.e., $\mathcal{A}$ is so; cf. Lemma 10.1) [and not $\sim$-classical {in particular, $\sim$-paraconsistent/$(\veebar,\sim)$-paracomplete}]. Then, (ii)$\Rightarrow$(iii)$\Rightarrow$(i)[$\Rightarrow$(ii){$\Leftrightarrow$(iv)$\Leftrightarrow$(v)}], where:*

(i) *C' is $\sim$-classical;*
(ii) *C' is proper, consistent and $\veebar$-disjunctive[{/ as well as non-pseudo-axiomatic}];*
(iii) *2 forms a subalgebra of $\mathfrak{A}$, C' being defined by $\mathcal{A}{\restriction}2$;*
(iv) *$C' = C^{\mathrm{R/EM}}$ is consistent;*
(v) *C' is consistent, $\veebar$-disjunctive and not $\sim$-paraconsistent/$(\veebar,\wr)$-paracomplete.*

In particular, any $\veebar$-disjunctive three-valued [non-]$\sim$-classical [{more specifically, $\sim$-paraconsistent/$(\veebar,\sim)$-paracomplete}] Σ-logic [with subclassical negation $\sim$] has no proper consistent $\veebar$-disjunctive (in particular, axiomatic) [non-$\sim$-classical {more specifically, $\sim$-paraconsistent/both $(\veebar,\sim)$-paracomplete and non-pseudo-axiomatic}] extension, any $\sim$-classical extension being a unique one and $\veebar$-disjunctive [{as well as relatively axiomatized by (10.1)/(2.11)*}].*

Proof. First, (i) is a particular case of (iii).

[Next, (i)$\Rightarrow$(ii) is by Lemma 10.1{/ and Remark 2.3}.]

Further, assume (ii) holds. Then, in case C is non-$(\veebar,\wr)$-paracomplete (in particular, either $\wr$-classical or $\wr$-paraconsistent), (2.11) is a theorem of it, and so of C', in which case this is non-pseudo-axiomatic. Hence, in any case, C' is non-pseudo-axiomatic. Therefore, by Remark 2.4 and Corollary 3.28, C' is defined by $\mathsf{S} \triangleq (\mathrm{Mod}(C') \cap \mathbf{S}^*_*(\mathcal{A}))$, in which case $\mathcal{A} \notin \mathsf{S} \neq \varnothing$. Consider any $\mathcal{B} \in \mathsf{S}$. Then, since $\mathcal{A}$ is false-/truth-singular, while $\mathcal{B}$ is consistent and truth-non-empty, we have $(0/1)_{\mathcal{A}} \in B$, in which case $(1/0)^{\sim}_{\mathcal{A}} = \sim^{\mathfrak{A}}(0/1)_{\mathcal{A}} \in B$, and so $(\frac{1}{2})^{\sim}_{\mathcal{A}} \notin B$, for $B \neq A$. Thus, $B = 2^{\sim}_{\mathcal{A}}$ forms a subalgebra of $\mathfrak{A}$, while $\mathsf{S} = \{\mathcal{B}\}$, so (iii) holds.

[{Now, assume (iii) holds. Then, $\mathcal{A}{\restriction}2$ is the only non-$\sim$-paraconsistent/non-$(\veebar,\sim)$-paracomplete consistent submatrix of $\mathcal{A}$. In this way, Theorem 3.26 and Remark 10.4 imply (iv).

Likewise, Theorem 3.26 and Remark 10.4 yield (iv)$\Rightarrow$(v).

Finally, (ii) is a particular case of (v)/, for any non-$(\veebar,\sim)$-paracomplete Σ-logic has the theorem (2.11), and so is non-pseudo-axiomatic.}]

At last, Theorem 4.1 and Corollary 2.10 complete the argument. □

In case C is Kleene's three-valued logic [4], that is both disjunctive and paracomplete as well as purely-inferential (unless it is garbled with its "bounded" expansion by constants 0 and 1, as it sometimes done in certain literature), Theorem 10.15 (more specifically, the fact that the non-$\sim$-classical [because it is distinct from C^{EM}] C^{EM}_{+0} is a proper consistent $\vee$-disjunctive extension of C) shows that the optional stipulation "non-pseudo-axiomatic" is essential for (ii)$\Rightarrow$(i) and the final assertion of Theorem 10.5 to hold.

Theorem 10.6. *[Providing C is non-$\sim$-classical] C has a [$\veebar$disjunctive] $\sim$-classical extension (viz., model [cf. Lemma 10.1]) if[f] 2 forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{A}{\restriction}2$ is isomorphic to any $\sim$-classical model of C, and so defines a unique $\sim$-classical extension of C.*

Proof. The "if"+"in which case" part is by Theorem 6.5. [Conversely, let $\mathcal{D}$ be a $\veebar$-disjunctive $\sim$-classical model of C. We prove that 2 forms a subalgebra of $\mathfrak{A}$ by contradiction. For suppose 2 does not form a subalgebra of $\mathfrak{A}$. Then, by Theorem 6.5, L_4 forms a subalgebra of $\mathfrak{A}^2$, while $\mathcal{A}$ is false-singular, whereas $\mathcal{B} \triangleq (\mathcal{A}^2{\restriction}L_4)$ is a strict surjective homomorphic counter-image of $\mathcal{D}$, in which case it is $\veebar$-disjunctive, for $\mathcal{D}$ is so. Therefore, as $\langle\frac{1}{2},1\rangle \in D^{\mathcal{B}}$, for $\mathcal{A}$ is false-singular, we have

$\{\langle\frac{1}{2},1\rangle \veebar^{\mathfrak{B}} \langle 0,\frac{1}{2}\rangle, \langle 0,\frac{1}{2}\rangle \veebar^{\mathfrak{B}} \langle\frac{1}{2},1\rangle\} \subseteq D^{\mathcal{B}}$, in which case we get $\{\frac{1}{2} \veebar^{\mathfrak{A}} 0, 0 \veebar^{\mathfrak{A}} \frac{1}{2}\} \subseteq D^{\mathcal{A}}$, and so we eventually get $(\langle 0,\frac{1}{2}\rangle \veebar^{\mathfrak{B}} \langle\frac{1}{2},0\rangle) \in D^{\mathcal{B}}$. This contradicts to the fact that $(\{\langle 0,\frac{1}{2}\rangle, \langle\frac{1}{2},0\rangle\} \cap D^{\mathcal{B}}) = \varnothing$, as required.] □

It is remarkable that the $\veebar$-disjunctivity of C is not required in the formulation of Theorem 10.6, making it the right algebraic criterion of C's being "genuinely subclassical" in the sense of having a *genuinely* (viz., functionally-complete) classical extension. And what is more, collectively with Lemma 10.1 and Corollary 10.2, it yields the following "disjunctive" analogue of Corollary 9.3:

Corollary 10.7. *Suppose $\mathcal{A}$ is $\veebar$-disjunctive (viz., C is so; cf. Lemma 10.1). Then, C is $\sim$-subclassical iff either of the following hold:*

(i) $\theta^{\mathcal{A}} \in \mathrm{Con}(\mathfrak{A})$*, in which case $\mathcal{A}/\theta^{\mathcal{A}}$ is isomorphic to any $\sim$-classical model of C, and so defines a unique $\sim$-classical extension of C, that is, C^{PC};*

(ii) 2 *forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{A}{\restriction}2$ is isomorphic to any $\sim$-classical model of C, and so defines a unique $\sim$-classical extension of C, that is, C^{PC}.*

Then, since $(\mathcal{A}{\restriction}\{\sim\}){\restriction}2$ is the only proper consistent submatrix of $\mathcal{A}{\restriction}\{\sim\}$, by Corollaries 10.2, 10.7 and Theorem 3.26, we also get:

Corollary 10.8. *Suppose $\mathcal{A}$ is both $(\veebar,\sim)$-paracomplete/$\sim$-paraconsistent and $\veebar$-disjunctive/$\sqsupset$-implicative (viz., C is so; cf. Lemma 10.1/10.3). Then, C has a proper consistent axiomatic extension iff it is $\sim$-subclassical, in which case C^{PC} is a unique proper consistent axiomatic extension of C and is relatively axiomatized by* (2.11)/(10.2).

This covers arbitrary three-valued expansions (cf. Corollary 5.9 in this connection) of Kleene's|[the implication-less fragment of]Gödel's three-valued logic [4]|[2]/both LA, HZ and P^1, subsuming Theorem 6.3 of [9].

Likewise, by Theorems 4.1, 7.3, 10.6 and Remarks 2.6(i)**a)** and 9.4, we get the following "disjunctive" analogue of Corollary 9.6, being essentially beyond the scopes of the reference [Pyn 95b] of [11], and so becoming a one more substantial advance of the present study with regard to that one:

Corollary 10.9. *Any [$\sim$-paraconsistent] three-valued Σ-logic having a $\veebar$-disjunctive $\sim$-classical extension (in particular, being both $\veebar$-disjunctive and $\sim$-subclassical; cf. Lemma 10.1) has no proper $\sim$-paraconsistent extension [and so is maximally so].*

On the other hand, as opposed to Corollary 9.6, the condition of being $\sim$-subclassical in the formulation of Corollary 10.9 is essential, as it follows from:

Example 10.10. Let $\Sigma = \{\sim[,\vee]\}$ as well as $\mathcal{A}$ is both false-singular and canonical, while $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$ [whereas: $(\vee^{\mathfrak{A}} = ((\pi_0{\restriction}\Delta_A) \cup ((A^2 \setminus \Delta_A) \times \{\frac{1}{2}\}))$ is commutative, in which case (2.3), (2.4) and (2.5) are true in $\mathcal{A}$, and so, by Lemma 10.1, C is $\vee$-disjunctive]. But, L_3 forms a subalgebra of $\mathfrak{A}^2$, so, by Theorem 7.3, C is not maximally $\sim$-paraconsistent [and so is not $\sim$-subclassical, by Corollary 10.9]. □

Finally, note that (2.11) is a theorem of C, whenever $\mathcal{A}$ is both false-singular and $\veebar$-disjunctive. In this way, by Corollaries 6.8, 8.5 and Theorem 8.1, we get the following "disjunctive" analogue of Corollary 9.5:

Corollary 10.11. *Suppose C is both $\veebar$-disjunctive (viz., $\mathcal{A}$ is so; cf. Lemma 10.1) and $\sim$-subclassical. Then, C has no [non-]inferentially consistent non-$\sim$-subclassical (viz, not being a sublogic of C^{PC}; cf. Lemma 2.9 and Theorem 6.5) extension [iff either $\mathcal{A}$ is false-singular or $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$].*

10.3. **Paracomplete extensions.**

Lemma 10.12. *Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}^2$ generated by $K_3 \triangleq \{\langle 0,0\rangle, \langle 1,1\rangle, \langle \frac{1}{2},1\rangle\}$, $\mathcal{B} \triangleq (\mathcal{A}^2{\restriction}B)$ and C' the logic of $\mathcal{B}$. Suppose C is both $\veebar$-disjunctive and $(\veebar,\sim)$-paracomplete (viz. $\mathcal{A}$ is so; cf. Lemma 10.1) as well as $\sim$-subclassical. Then, C' is a non-pseudo-axiomatic $(\veebar,\sim)$-paracomplete extension of C and is a proper sublogic of C^{PC}. Moreover, (i)⇒(ii)⇔(iii)⇔(iv)⇔(v)⇒(vi), where:*

(i) *$\mathcal{A}$ is implicative;*
(ii) *$\langle 1,0\rangle \in B$;*
(iii) *$B \nsubseteq K_4 \triangleq (K_3 \cup \{\langle \frac{1}{2},0\rangle\})$;*
(iv) *neither K_3 nor K_4 forms a subalgebra of $\mathfrak{A}^2$;*
(v) *$C' \neq C$;*
(vi) *$\mathfrak{A}$ is not regular.*

Proof. Since any $\sim$-classical $\veebar$-disjunctive Σ-logic is not $(\veebar,\sim)$-paracomplete, in that case, $\mathcal{A}$ is truth-singular, while C is not $\sim$-classical, and so, by Theorem 6.5, 2 forms a subalgebra of $\mathfrak{A}$, while C^{PC} is defined by the $\veebar$-disjunctive $\sim$-classical (and so non-$(\veebar,\sim)$-paracomplete) Σ-matrix $\mathcal{A}{\restriction}2$, whereas $D^{\mathcal{B}} = \{\langle 1,1\rangle\} \neq B \supseteq K_3 \ni \langle 0,0\rangle \neq \langle 1,1\rangle$, and so, by (2.13) and Remark 2.3, C' is a non-pseudo-axiomatic consistent extension of C, in which case it is inferentially consistent, and so, by Theorem 8.1, C' is a sublogic of C^{PC}. And what is more, as $\pi_0[K_3] = A$, $(\pi_0{\restriction}B) \in \hom^{\mathrm{S}}(\mathcal{B},\mathcal{A})$, in which case, by (2.14), $\mathcal{B}$ is $(\veebar,\sim)$-paracomplete, for $\mathcal{A}$ is so, and so is C', being thus distinct from C^{PC}.

Next, assume $\mathcal{A}$ is $\sqsupset$-implicative, where $\sqsupset$ is a (possibly, secondary) binary connective of Σ, in which case, since $D^{\mathcal{A}} = \{1\}$, $(\frac{1}{2} \sqsupset^{\mathfrak{A}} 0) = 1$ and, as 2 forms a subalgebra of $\mathfrak{A}$, $(1 \sqsupset^{\mathfrak{A}} 0) = 0$, and so $\langle 1,0\rangle = (\langle \frac{1}{2},1\rangle \sqsupset^{\mathfrak{A}^2} \langle 0,0\rangle) \in B$, for $\{\langle \frac{1}{2},1\rangle, \langle 0,0\rangle\} \subseteq K_3 \subseteq B$. Thus, (i)⇒(ii) holds.

Further, (ii)⇒(iii) is by the fact that $\langle 1,0\rangle \notin K_4$. The converse is by the fact that $\sim^{\mathfrak{A}^2}\langle 0,1\rangle = \langle 1,0\rangle$, while $K_4 = ((A \times 2) \setminus \{\langle 0,1\rangle, \langle 1,0\rangle\})$, whereas $\pi_1[K_3] = 2$ forms a subalgebra of $\mathfrak{A}$, in which case $\pi_1[B] = 2$, and so $B \subseteq (A \times 2)$. Furthermore, (iii)⇒(iv) is by the inclusion $K_3 \subseteq K_4$. The converse is by the fact that any singleton has no proper non-empty subset, while $K_3 \subseteq B$.

Now, assume $\mathfrak{A}$ is regular, while (ii) holds. Then, there is some $\varphi \in \mathrm{Fm}^3_\Sigma$ such that $\varphi^{\mathfrak{A}}(0,1,\frac{1}{2}) = 1$ and $\varphi^{\mathfrak{A}}(0,1,1) = 0$. On the other hand, we have $\frac{1}{2} \sqsubseteq 1$, in which case, by the regularity/reflexivity of $\mathfrak{A}/\sqsubseteq$, we get $1 \sqsubseteq 0$, and so this contradiction shows that (ii)⇒(iv) holds.

Finally, assume (ii) holds. We prove that $C' \neq C$, by contradiction. For suppose $C' = C$, in which case $\mathcal{A}$ is a finite consistent truth-non-empty $\veebar$-disjunctive simple (in view of Theorem 5.3) model of $C' \supseteq C$, being, in its turn, weakly $\veebar$-disjunctive, and so being $\mathcal{B}$. Then, by Lemmas 2.7, 2.8 and Remark 2.5, there is some truth-non-empty submatrix $\mathcal{D}$ of $\mathcal{B}$, being a strict surjective homomorphic counter-image of $\mathcal{A}$, in which case it is both truth-non-empty, $(\veebar,\sim)$-paracomplete and $\veebar$-disjunctive, for $\mathcal{A}$ is so, and so $D^{\mathcal{D}} = \{\langle 1,1\rangle\}$, while there is some $a \in D$ such that $D \in b \triangleq (a \veebar^{\mathfrak{A}^2} \sim^{\mathfrak{A}^2} a) \notin D^{\mathcal{D}} = \{\langle 1,1\rangle\}$. On the other hand, since $\pi_1[K_3] = 2$ forms a subalgebra of $\mathfrak{A}$, in which case $\pi_1[D] \subseteq \pi_1[B] \subseteq 2$, by the $\veebar$-disjunctivity of $\mathcal{A}$, we have $\pi_1(b) = 1$, in which case $\pi_0(b) \neq 1$, and so we have the following two exhaustive cases:

- $\pi_0(a) = \frac{1}{2}$.
  Then, as $\langle 0,0\rangle = \sim^{\mathfrak{A}^2}\langle 1,1\rangle \in D$, we have $K_3 \subseteq D$, in which case we get $\langle 1,0\rangle \in D$, and so $\langle 0,1\rangle = \sim^{\mathfrak{A}^2}\langle 1,0\rangle \in D$.
- $\pi_0(a) = 0$.
  Then, we also have $\langle 1,0\rangle = \sim^{\mathfrak{A}^2}\langle 0,1\rangle \in D$.

Thus, anyway, $\{\langle 0,1\rangle,\langle 1,0\rangle\}\subseteq(D\setminus D^{\mathcal{D}})$, while, by the $\veebar$-disjunctivity of $\mathcal{A}$, $(\langle 0,1\rangle\veebar^{\mathfrak{A}^2}\langle 1,0\rangle)=\langle 1,1\rangle\in D^{\mathcal{D}}$. This contradicts to the $\veebar$-disjunctivity of $\mathcal{D}$. Thus, (v) holds. Conversely, assume $\langle 1,0\rangle\notin B$, in which case $(\pi_0{\upharpoonright}B)\in\hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B},\mathcal{A})$, and so $C'=C$, by (2.13), as required. □

Lemma 10.13. *Suppose C is both $\veebar$-disjunctive (viz. $\mathcal{A}$ is so; cf. Lemma 10.1) and $\sim$-subclassical, while either K_3 or K_4 forms a subalgebra of $\mathfrak{A}^2$. Then, C has no proper $(\veebar,\sim)$-paracomplete non-pseudo-axiomatic extension.*

Proof. Let C' be a $(\veebar,\sim)$-paracomplete non-pseudo-axiomatic extension of C, in which case $(x_1\veebar\sim x_1)\notin T\triangleq C'(x_0)\ni x_0$, while, by the structurality of C', $\langle\mathfrak{Fm}^{\omega}_{\Sigma},T\rangle$ is a model of C' (and so of C), and so is its $(\veebar,\sim)$-paracomplete (and so consistent) truth-non-empty finitely-generated submatrix $\mathcal{B}\triangleq\langle\mathfrak{Fm}^2_{\Sigma},\mathrm{Fm}^2_{\Sigma}\cap T\rangle$, in view of (2.13), whereas C is $(\veebar,\sim)$-paracomplete (viz., $\mathcal{A}$ is so), in which case it is not $\sim$-classical, and so, by Corollaries 10.2 and 10.7, 2 forms a subalgebra of $\mathfrak{A}$. Then, since $\mathcal{A}$ is $\veebar$-disjunctive, and so, being $(\veebar,\sim)$-paracomplete, is truth-singular, we have $((1/0)\veebar^{\mathfrak{A}}(0/1))=1$, in which case we get $((1/0)\veebar^{\mathfrak{A}}\sim^{\mathfrak{A}}(1/0))=1$, and so $(\frac{1}{2}\veebar^{\mathfrak{A}}\sim^{\mathfrak{A}}\frac{1}{2})=\frac{1}{2}$, for, otherwise, as $(\{\frac{1}{2},\sim^{\mathfrak{A}}\frac{1}{2}\}\cap D^{\mathcal{A}})=\varnothing$, we would have $(\frac{1}{2}\veebar^{\mathfrak{A}}\sim^{\mathfrak{A}}\frac{1}{2})=0$, in which case we would get $(\langle\frac{1}{2},1\rangle\veebar^{\mathfrak{A}^2}\sim^{\mathfrak{A}^2}\langle\frac{1}{2},1\rangle)=\langle 0,1\rangle\notin K_4\supseteq K_3$, and so neither $K_3\ni\langle\frac{1}{2},1\rangle$ nor K_4 would form a subalgebra of $\mathfrak{A}^2$.

Further, by Lemma 2.8, there are some set I, some $\overline{\mathcal{C}}\in\mathbf{S}(\mathcal{A})^I$ and some subdirect product $\mathcal{D}$ of it, being a strict homomorphic counter-image of a strict homomorphic image of $\mathcal{B}$, and so a $(\veebar,\sim)$-paracomplete (in particular, consistent, in which case $I\neq\varnothing$), truth-non-empty model of C', in view of (2.13), for $\mathcal{B}$ is so. Take any $a\in D^{\mathcal{D}}\neq\varnothing$, in which case $D\ni a=(I\times\{1\})$, and so $D\ni b\triangleq\sim^{\mathfrak{D}}a=(I\times\{0\})$. Moreover, there is some $c\in D$ such that, since $((1/0/\frac{1}{2})\veebar^{\mathfrak{A}}\sim^{\mathfrak{A}}(1/0/\frac{1}{2}))=(1/1/\frac{1}{2})$, $(D\cap\{\frac{1}{2},1\}^I)\ni d\triangleq(c\veebar^{\mathfrak{D}}\sim^{\mathfrak{D}}c)\notin D^{\mathcal{D}}$, in which case $J\triangleq\{i\in I\mid\pi_i(d)=\frac{1}{2}\}\neq\varnothing$. Given any $\bar{e}\in A^2$, set $(e_0\wr e_1)\triangleq((J\times\{e_0\})\cup((I\setminus J)\times\{e_1\}))$. In this way, $D\ni a=(1\wr 1)$, $D\ni b=(0\wr 0)$ and $D\ni d=(\frac{1}{2}\wr 1)$. Consider the following complementary cases:

- $J=I$,
  in which case, as $I\neq\varnothing$, $\{\langle e,I\times\{e\}\rangle\mid e\in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, and so $C'\subseteq C$, by (2.13).
- $J\neq I$,
  Let $\mathfrak{E}$ be the subalgebra of $\mathfrak{A}^2$ generated by K_3 and $\mathcal{E}\triangleq(\mathcal{A}^2{\upharpoonright}E)$. Then, as $J\neq\varnothing\neq(I\setminus J)$ and $\{(x\wr y)\mid\langle x,y\rangle\in K_3\}\subseteq D$, $\{\langle\langle x,y\rangle,(x\wr y)\rangle\mid\langle x,y\rangle\in E\}$ is an embedding of $\mathcal{E}$ into $\mathcal{D}$. Hence, $C'\subseteq C$, by (2.13) and Lemma 10.12.

□

By Remarks 2.2, 2.3, Lemmas 10.12, 10.13 and Corollaries 6.7, 10.7 and 10.8, we immediately have:

Theorem 10.14. *Suppose C is $\veebar$-disjunctive and $(\veebar,\sim)$-paracomplete (viz., $\mathcal{A}$ is so; cf. Lemma 10.1). Then, C has no proper $(\veebar,\sim)$-paracomplete [non-pseudo-]axiomatic extension (i.e, C is maximally [non-]axiomatically inferentially $(\veebar,\sim)$-paracomplete) [iff either $\{0,1\}$ does not form a subalgebra of $\mathfrak{A}$ or either K_3 or K_4 forms a subalgebra of $\mathfrak{A}^2$].*

Likewise, by Remarks 2.2, 2.3, 2.4, Lemmas 10.1, 10.12, 10.13, Corollaries 6.7, 10.2, 6.8, 10.7, 10.8 and Theorem 8.1, we also get:

Theorem 10.15. *Suppose C is both $\veebar$-disjunctive, $(\veebar,\sim)$-paracomplete and [not] $\sim$-subclassical as well as has a/no theorem. Then, proper (arbitrary/merely non-pseudo-axiomatic) extensions of C form the four-element diamond (resp., two-element chain) [resp., $(2(-1))$-element chain] depicted at Figure 1 (with merely*

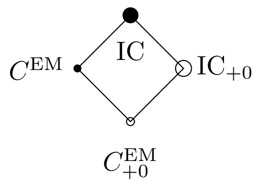

FIGURE 1. The lattice of proper extensions of C.

solid circles) [(and) with solely big circles] iff either C is not $\sim$-subclassical or either K_3 or K_4 forms a subalgebra of $\mathfrak{A}^2$ {in particular, $\mathfrak{A}$ is regular; cf. Lemma 10.12}, $\mathrm{IC}_{\langle/+0\rangle}\,|C^{\mathrm{EM}}_{\langle/+0\rangle}$ being $\underline{\vee}$-disjunctive, relatively axiomatized by $(\langle x_0 \vdash\rangle(x_1|(x_1 \underline{\vee} {\sim} x_1))$ and defined by $(\varnothing|\{\mathcal{A}\restriction 2\})\langle\cup\{\mathcal{A}\restriction\{\frac{1}{2}\}\}\rangle$, respectively.

Perhaps, most representative instances of this subsection are three-valued expansions (by constants, as regular ones and with $K_{4[-1]}$ [not] forming a subalgebra of $\mathfrak{A}^2$) of Kleene' logic [4], {the implication-free fragment of} Gödel's one [2] — as non-regular (because of negation) ones but with $K_{3[+1]}$ [not] forming a subalgebra of $\mathfrak{A}^2$ — and Łukasiewicz' one [6] (as an implicative one), having a unique proper non-pseudo-axiomatic $(\underline{\vee}, \sim)$-paracomplete extension (cf. [17]).

11. SELF-EXTENSIONALITY

In case C is $\sim$-classical, it is self-extensional, in view of Example 3.10. Here, we mainly explore the opposite case.

First, we have the *dual* three-valued $\sim$-super-classical Σ-matrix $\partial(\mathcal{A}) \triangleq \langle\mathfrak{A}, \{1\}\cup(\{\frac{1}{2}\}\cap(A \setminus D^{\mathcal{A}}))\rangle$, in which case it is false/truth-singular iff $\mathcal{A}$ is not so, while:

$$(\theta^{\mathcal{A}} \cap \theta^{\partial(\mathcal{A})}) = \Delta_A. \tag{11.1}$$

Likewise, set $\mathcal{A}_{a[+(b)]} \triangleq \langle\mathfrak{A}, \{[\frac{1}{2}(-\frac{1}{2}+b),]a\}\rangle$, where $a[(,b)] \in A$, in which case $(\partial(\mathcal{A})/\mathcal{A}) = \mathcal{A}_{1[+]}$, whenever $\mathcal{A}$ is [not] false-/truth-singular, while:

$$(\theta^{\mathcal{A}_{i[+]}} \cap \theta^{\mathcal{A}_{(1-i)[+]}}) = \Delta_A, \tag{11.2}$$

for all $i \in 2$.

Further, given any $i \in 2$, put $h_i \triangleq (\Delta_2 \cup \{\langle\frac{1}{2}, i\rangle\}) : (3 \div 2) \to 2$, in which case:

$$h^{-1}_{0/1}[D^{\mathcal{A}}] = D^{\partial(\mathcal{A})}, \tag{11.3}$$

whenever $\mathcal{A}$ is false-/truth-singular.

Finally, let $h_{1-} : (3 \div 2) \to (3 \div 2), a \mapsto (1-a)$, in which case:

$$h^{-1}_{1-}[D^{\mathcal{A}_{i[+]}}] = D^{\mathcal{A}_{(1-i)[+]}}, \tag{11.4}$$

for all $i \in 2$.

11.1. **Conjunctive logics.** Below, we use tacitly the following preliminary observation:

Remark 11.1. Suppose C is $\overline{\wedge}$-conjunctive, non-$\sim$-classical (in which case $\mathcal{A}$ is simple; cf. Theorem 5.3) and self-extensional. Then, by Corollary 3.12(i)$\Rightarrow$(ii), $\mathfrak{A}$, being finite, is a $\overline{\wedge}$-semilattice with $\flat^{\mathfrak{A}}_{\overline{\wedge}}$, in which case, as $0 \notin D^{\mathcal{A}}$, by the $\overline{\wedge}$-conjunctivity of $\mathcal{A}$, we have $\flat^{\mathfrak{A}}_{\overline{\wedge}} = (\flat^{\mathfrak{A}}_{\overline{\wedge}} \,\overline{\wedge}^{\mathfrak{A}}\, 0) \notin D^{\mathcal{A}}$. □

Lemma 11.2. *Suppose C is $\overline{\wedge}$-conjunctive, non-$\sim$-classical (in which case $\mathcal{A}$ is simple; cf. Theorem 5.3) and self-extensional. Then,*

$$\frac{1}{2} \leq^{\mathfrak{A}}_{\overline{\wedge}} 1. \tag{11.5}$$

Moreover, the following are equivalent:

(i) $\flat^{\mathfrak{A}}_{\overline{\wedge}} = 0$ *(in particular, $\mathcal{A}$ is false-singular);*
(ii) $\flat^{\mathfrak{A}}_{\overline{\wedge}} \neq \frac{1}{2}$*;*
(iii) $0 \leq^{\mathfrak{A}}_{\overline{\wedge}} 1$*;*
(iv) $0 \leq^{\mathfrak{A}}_{\overline{\wedge}} \frac{1}{2}$*;*
(v) 2 *forms a subalgebra of* $\mathfrak{A}$*;*
(vi) $h_{1-} \notin \hom(\mathfrak{A}, \mathfrak{A})$*;*
(vii) $h_{0/1} \in \hom(\mathfrak{A}, \mathfrak{A})$*, whenever $\mathcal{A}$ is false-/truth-singular.*

Proof. First, we prove (11.5) by contradiction. For suppose $\frac{1}{2} \not\leq^{\mathfrak{A}}_{\overline{\wedge}} 1$, in which case $\flat^{\mathfrak{A}}_{\overline{\wedge}} \neq \frac{1}{2}$, and so $\frac{1}{2} \not\leq^{\mathfrak{A}}_{\overline{\wedge}} \flat^{\mathfrak{A}}_{\overline{\wedge}} = 0$. Then, $\mathcal{A}_{\frac{1}{2}}$ is $\overline{\wedge}$-conjunctive, and so, being truth-non-empty, is a model of C, by Corollary 3.12, in which case, by Lemmas 2.7 and 2.8, there are some non-empty finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$, some subdirect product $\mathcal{D}$ of it, some Σ-matrix $\mathcal{E}$, some $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}_{\frac{1}{2}}, \mathcal{E})$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{E})$, in which case $\mathcal{D}$ is truth-non-empty, for $\mathcal{A}_{\frac{1}{2}}$ is so, and so, by the following claim, $\{I \times \{c\} \mid c \in 2\} \subseteq D$:

Claim 11.3. *Let I be a finite set, $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and $\mathcal{D}$ a truth-non-empty subdirect product of it. Then, $\{I \times \{c\} \mid c \in 2\} \subseteq D$.*

Proof. Consider the following complementary cases:

- $\mathcal{A}$ is truth-singular,
  and so is $\mathcal{D}$, being also truth-non-empty, in which case $a \triangleq (I \times \{1\}) \in D^{\mathcal{D}}$, and so $D \ni b \triangleq {\sim}^{\mathfrak{D}} a = (I \times \{0\})$.
- $\mathcal{A}$ is false-singular.
  Then, by Lemma 3.1, we have $b \triangleq (I \times \{0\}) \in D$, and so $D \ni a \triangleq {\sim}^{\mathfrak{D}} b = (I \times \{1\})$. ☐

Given any Σ-matrix $\mathcal{H}$, set $\mathcal{H}' \triangleq (\mathcal{H} \upharpoonright \{\sim\})$. In this way, $\mathcal{D}'$ is a submatrix of $(\mathcal{A}')^I$, while $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}'_{\frac{1}{2}}, \mathcal{E}')$, whereas $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}', \mathcal{E}')$. And what is more, 2 forms a subalgebra of $\mathfrak{A}'$. Then, as $I \neq \varnothing$, $e \triangleq \{\langle c, I \times \{c\}\rangle \mid c \in 2\}$ is an embedding of $\mathcal{C} \triangleq (\mathcal{A}' \upharpoonright 2)$ into $\mathcal{D}'$, in which case $f \triangleq (g \circ e) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{C}, \mathcal{E}')$ is injective, by Remark 2.5, for $\mathcal{C}$, being $\sim$-classical, is simple. Hence, $F \triangleq (\operatorname{img} f)$ forms a subalgebra of $\mathfrak{E}'$, in which case f is an isomorphism from $\mathcal{C}$ onto $\mathcal{F} \triangleq (\mathcal{E}' \upharpoonright F)$, and so $\mathcal{F}$ is $\sim$-classical, for $\mathcal{C}$ is so. Then, $G \triangleq h^{-1}[F]$ forms a subalgebra of $\mathfrak{A}'$, in which case $h \upharpoonright G$ is a strict surjective homomorphism from $\mathcal{G} \triangleq (\mathcal{A}'_{\frac{1}{2}} \upharpoonright G)$ onto $\mathcal{F}$, and so $\mathcal{G}$ is both truth-non-empty and $\sim$-negative, for $\mathcal{F}$, being $\sim$-classical, is so, as well as truth-singular, for $\mathcal{A}'_{\frac{1}{2}}$ is so. Therefore, $D^{\mathcal{G}} = \{\frac{1}{2}\}$, in which case ${\sim}^{\mathfrak{A}} \frac{1}{2} \in (G \setminus D^{\mathcal{G}}) = (2 \cap G)$, and so ${\sim}^{\mathfrak{A}} {\sim}^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$. This contradicts to the fact that ${\sim}^{\mathfrak{A}}[2] \subseteq 2 \not\ni \frac{1}{2}$, in which case (11.5) holds, and so does (iv)⇒(iii).

Next, (i)⇔(ii) is immediate, while (iv) is a particular case of (i). Conversely, if (iii) did hold but (ii) did not so, in which case the $\sim$-paraconsistent (in particular, truth-non-empty) Σ-matrix $\mathcal{A}_{1+0}$ was $\overline{\wedge}$-conjunctive, and so, by Corollary 3.12, was a model of C, then C would be $\sim$-paraconsistent, that is, $\mathcal{A}$ would be so, in which case this would be false-singular, and so (i) would hold. Therefore, (iii)⇒(i) holds. Thus, we have proved that (i,ii,iii,iv) are equivalent to one another.

Further, by the $\overline{\wedge}$-conjunctivity of $\mathcal{A}$ and the fact that $0 \notin D^{\mathcal{A}} \ni 1$, we have:

$$(1 \mathbin{\overline{\wedge}^{\mathfrak{A}}} 0) \neq 1. \tag{11.6}$$

Therefore, if (iii) does not hold, that is, $(1 \mathbin{\overline{\wedge}^{\mathfrak{A}}} 0) \neq 0$, then, by (11.6), $(1 \mathbin{\overline{\wedge}^{\mathfrak{A}}} 0) = \frac{1}{2}$, in which case (v) does not hold, and so (v)⇒(iii) holds. Conversely, assume (i) holds. We prove (v) by contradiction. For suppose 2 does not form a subalgebra of $\mathfrak{A}$. Then, there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(0, 1) = \frac{1}{2}$. Moreover, by (11.5) and (i), $\partial(\mathcal{A})$ is $\bar{\wedge}$-conjunctive, in which case, by Corollary 3.12, it, being truth-non-empty, is a model of C, and so, by Lemmas 2.7 and 2.8, there are some non-empty finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$, some subdirect product $\mathcal{D}$ of it, some Σ-matrix $\mathcal{E}$, some $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\partial(\mathcal{A}), \mathcal{E})$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{E})$, in which case $\mathcal{D}$ is truth-non-empty, for $\partial(\mathcal{A})$ is so, and so, by Claim 11.3, $(a/b) \triangleq (I \times \{0/1\}) \in D$. Then, $D \ni \varphi^{\mathfrak{D}}(a,b) = (I \times \{\frac{1}{2}\}$, in which case, as $I \neq \varnothing$, $e \triangleq \{\langle c, I \times \{c\}\rangle \mid c \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, and so $g \circ e$ is that into $\mathcal{E}$, in view of Remark 2.5. In this way, $\mathcal{A}$ is false-/truth-singular, whenever $\partial(\mathcal{A})$ is so. This contradiction shows that (v) holds. Thus, (i,ii,iii,iv,v) are equivalent.

Now, assume (vi) does not hold. In that case, if (iii) did hold, then we would have $1 = h_{1-}(0) = h_{1-}(0 \mathbin{\bar{\wedge}^{\mathfrak{A}}} 1) = (h_{1-}(0) \mathbin{\bar{\wedge}^{\mathfrak{A}}} h_{1-}(1)) = (1 \mathbin{\bar{\wedge}^{\mathfrak{A}}} 0) = 0$. Therefore, (iii)$\Rightarrow$(vi) holds. Conversely, assume (i,ii,iii,iv,v) do not hold. In particular, there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(0,1) = \frac{1}{2}$. Moreover, $\mathcal{A}_0$ is then $\bar{\wedge}$-conjunctive, and so, being truth-non-empty, is a model of C, by Corollary 3.12, in which case, by Lemmas 2.7 and 2.8, there are some non-empty finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$, some subdirect product $\mathcal{D}$ of it, some Σ-matrix $\mathcal{E}$, some $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}_0, \mathcal{E})$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{E})$, in which case $\mathcal{D}$ is truth-non-empty, for $\mathcal{A}_0$ is so, and so, by Claim 11.3, $(a/b) \triangleq (I \times \{0/1\}) \in D$, in which case $D \ni \varphi^{\mathfrak{D}}(a,b) = (I \times \{\frac{1}{2}\})$. Hence, as $I \neq \varnothing$, $e \triangleq \{\langle c, I \times \{c\}\rangle \mid c \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case $f \triangleq (g \circ e)$ is that into $\mathcal{E}$, by Remark 2.5, and so $3 = |A| \leqslant |E| \leqslant |A| = 3$. Therefore, $|E| = 3$, in which case h is injective, while $(\mathrm{img}\, f) = E$, and so $i \triangleq (h^{-1} \circ f)$ is an isomorphism from $\mathcal{A} = \mathcal{A}_1$ onto $\mathcal{A}_0$. In this way, since $D^{\mathcal{A}_d} = \{d\}$, for all $d \in A$, we have $i(1) = 0$, in which case we get $i(0) = i(\sim^{\mathfrak{A}} 1) = \sim^{\mathfrak{A}} i(1) = \sim^{\mathfrak{A}} 0 = 1$, and so $i(\frac{1}{2}) = \frac{1}{2}$. Thus, $\hom(\mathfrak{A}, \mathfrak{A}) \ni i = h_{1-}$, in which case (vi) does not hold, and so (i,ii,iii,iv,v,vi) are equivalent.

Finally, assume (vii) holds. Then, in case $\mathcal{A}$ is false-singular, (i) holds. Otherwise, $h_1 \in \hom(\mathfrak{A}, \mathfrak{A})$, in which case, if (ii) did not hold, then we would have $(\frac{1}{2} \mathbin{\bar{\wedge}^{\mathfrak{A}}} 0) = \frac{1}{2}$, and so we would get $1 = h_1(\frac{1}{2}) = h_1(\frac{1}{2} \mathbin{\bar{\wedge}^{\mathfrak{A}}} 0) = (h_1(\frac{1}{2}) \mathbin{\bar{\wedge}^{\mathfrak{A}}} h_1(0)) = (1 \mathbin{\bar{\wedge}^{\mathfrak{A}}} 0) \neq 1$, by (11.6). Therefore, anyway, (i,ii,iii,iv,v,vi) hold. Conversely, assume (i,ii,iii,iv,v,vi) hold. Then, by (v), 2 forms a subalgebra $\mathfrak{A}$, while, by (11.5) and (i), $\partial(\mathcal{A})$ is $\bar{\wedge}$-conjunctive, and so, being truth-non-empty, is a model of C, by Corollary 3.12. Consider the following complementary cases:

- $\mathcal{A}$ is false-singular.
  Consider the following complementary subcases:
  - $\sim^{\mathfrak{A}} \frac{1}{2} = 0$.
    Then, $\mathcal{A}$ is $\sim$-negative, in which case, by Remark 2.6(i)**a)**, it, being $\bar{\wedge}$-conjunctive, is $\bar{\wedge}^{\sim}$-disjunctive, and so, by Corollary 3.12, $\mathfrak{A}$ is a distributive $(\bar{\wedge}, \bar{\wedge}^{\sim})$-lattice, in which case $\flat^{\mathfrak{A}}_{\bar{\wedge}^{\sim}} = 1$, and so $\partial(\mathcal{A}) = \mathcal{A}_1$ is $\bar{\wedge}^{\sim}$-disjunctive, for $0 \leq^{\mathfrak{A}}_{\bar{\wedge}} \frac{1}{2} \leq^{\mathfrak{A}}_{\bar{\wedge}} 1$, by (11.5) and (i). Hence, by Lemmas 2.7, 2.8, 5.1 and Remark 2.5, there is some $h \in \hom_{\mathrm{S}}(\partial(\mathcal{A}), \mathcal{A})$. Then, as $\mathcal{A}$ is false-singular, $h[\{\frac{1}{2}, 0\}] = h[A \setminus d^{\partial(\mathcal{A})}] \subseteq (A \setminus D^{\mathcal{A}}) = \{0\}$, in which case $h(1) = h(\sim^{\mathfrak{A}} 0) = \sim^{\mathfrak{A}} h(0) = \sim^{\mathfrak{A}} 0 = 1$, and so $\hom(\mathfrak{A}, \mathfrak{A}) \ni h = h_0$.
  - $\sim^{\mathfrak{A}} \frac{1}{2} \neq 0$.
    Then, by (11.5), $\frac{1}{2} \leq^{\mathfrak{A}}_{\bar{\wedge}} \sim^{\mathfrak{A}} \frac{1}{2}$, in which case $\frac{1}{2} = (\frac{1}{2} \mathbin{\bar{\wedge}^{\mathfrak{A}}} \sim^{\mathfrak{A}} \frac{1}{2})$, and so $\sim^{\mathfrak{A}}(\frac{1}{2} \mathbin{\bar{\wedge}^{\mathfrak{A}}} \sim^{\mathfrak{A}} \frac{1}{2}) = \sim^{\mathfrak{A}} \frac{1}{2} \in D^{\mathcal{A}}$. Likewise, $\sim^{\mathfrak{A}}(i \mathbin{\bar{\wedge}^{\mathfrak{A}}} \sim^{\mathfrak{A}} i) = 1 \in D^{\mathcal{A}}$, for all $i \in 2$. Hence, $\sim(x_j \mathbin{\bar{\wedge}} \sim x_j) \in C(\varnothing)$, for each $j \in 2$. Therefore, by Lemma 3.19, $\sim^{\mathfrak{A}} \frac{1}{2} = \sim^{\mathfrak{A}}(\frac{1}{2} \mathbin{\bar{\wedge}^{\mathfrak{A}}} \sim^{\mathfrak{A}} \frac{1}{2}) = \sim^{\mathfrak{A}}(1 \mathbin{\bar{\wedge}^{\mathfrak{A}}} \sim^{\mathfrak{A}} 1) = 1$. Then, $\partial(\mathcal{A}) = \mathcal{A}_1$ is $\sim$-negative, in which case, by Remark 2.6(i)**a)**, it, being $\bar{\wedge}$-conjunctive, is $\bar{\wedge}^{\sim}$-disjunctive, and so $\sqsupset$-implicative, where

$(x_0 \sqsupset x_1) \triangleq (\sim x_0 \mathbin{\overline{\wedge}^{\sim}} x_1)$. Consider the following complementary subsubcases:

- $\partial(\mathcal{A})$ is not simple.
 Then, by Lemma 5.1, there are some $\sim$-classical Σ-matrix $\mathcal{B}$ and some $e \in \hom_{\mathrm{S}}^{\mathrm{S}}(\partial(\mathcal{A}), \mathcal{B})$. Therefore, by (2.13) and Theorem 6.5, there is some isomorphism i from $\mathcal{B}$ onto $\mathcal{A}{\restriction}2$, in which case $h \triangleq (i \circ e) \in \hom_{\mathrm{S}}(\partial(\mathcal{A}), \mathcal{A}{\restriction}2)$, and so $\hom(\mathfrak{A}, \mathfrak{A}) \ni h = h_0$.
- $\partial(\mathcal{A})$ is simple.
 Then, by Lemma 5.1, $\partial(\mathcal{A})$ is hereditarily simple, in which case, by Corollary 3.5, it has a unary binary equality determinant ϵ, and so $\varepsilon \triangleq \{\phi \sqsupset \psi \mid (\phi \vdash \psi) \in \epsilon\}$ is an axiomatic binary equality determinant for it. Moreover, $\mathcal{C} \triangleq (\mathcal{A}{\restriction}2) = (\partial(\mathcal{A}){\restriction}2)$, and so, by Lemma 3.4, ε is an equality determinant for $\mathcal{C}$ too. And what is more, by Lemmas 2.7, 2.8 and Remark 2.5, there are some non-empty set I, some submatrix $\mathcal{D}$ of $\mathcal{A}^I$ and some $g \in \hom(\mathcal{D}, \partial(\mathcal{A}))$. Then, as $\frac{1}{2} \in (A \setminus D^{\partial(\mathcal{A})})$, there is some $a \in (D \setminus D^{\mathcal{D}})$ such that $g(a) = \frac{1}{2}$. On the other hand, $\sim^{\mathfrak{A}}\frac{1}{2} = 1 \in D^{\partial(\mathcal{A})}$, in which case $b \triangleq \sim^{\mathfrak{D}} a \in D^{\mathcal{D}}$, and so $a \in \{\frac{1}{2}, 0\}^I$. Let $J \triangleq \{i \in I \mid \pi_i(a) = \frac{1}{2}\} \neq I$, for $a \notin D^{\mathcal{D}}$, while $\frac{1}{2} \in D^{\mathcal{A}}$. Given any $\bar{d} \in A^2$, set $(d_0 \wr d_1) \triangleq ((J \times \{d_0\}) \cup ((I \setminus J) \times \{d_1\})) \in A^I$, in which case $a = (\frac{1}{2} \wr 0)$, and so $b = (1 \wr 1)$. Let us prove, by contradiction, that $J \neq \varnothing$. For suppose $J = \varnothing$. Then, $(I \times \{1\} = b \in D \ni a = (I \times \{0\}$, in which case, as $I \neq \varnothing$, $e \triangleq \{\langle c, I \times \{c\}\rangle \mid c \in 2\}$ is an embedding of $\mathcal{C}$ into $\mathcal{D}$, and so $f \triangleq (g \circ e)$ is that into $\partial(\mathcal{A})$. In that case, $E \triangleq (\operatorname{img} f)$ forms a subalgebra of $\mathfrak{A}$. On the other hand, $a \in (\operatorname{img} e)$, in which case $\frac{1}{2} = g(a) \in E$, and so $E = A$, for $\mathfrak{A}$ is generated by $\{\frac{1}{2}\}$, because $(\sim^{\mathfrak{A}})^{2-j}\frac{1}{2} = j$, for all $j \in 2$. Thus, f is an isomorphism from $\mathcal{C}$ onto $\partial(\mathcal{A})$. This contradicts to the fact that $|C| = 2 \neq 3 = |A|$. Therefore, $J \neq \varnothing$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}^2$ generated by $\{\langle\frac{1}{2}, 0\rangle\}$. Then, as $J \neq \varnothing \neq (I \setminus J)$ and $(\frac{1}{2} \wr 0) = a \in D$, $e' \triangleq \{\langle\langle c, d\rangle, (c \wr d)\rangle \mid \langle c, d\rangle \in B\}$ is an embedding of $\mathcal{B} \triangleq (\mathcal{A}^2{\restriction}B)$ into $\mathcal{D}$, in which case $f' \triangleq (g \circ e') \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{B}, \partial(\mathcal{A}))$, for $f'[\{\langle\frac{1}{2}, 0\rangle\}] = g[\{a\}] = \{\frac{1}{2}\}$ generates $\mathfrak{A}$. Moreover, $g' \triangleq (\pi_1{\restriction}B) \in \hom^{\mathrm{S}}(\mathcal{B}, \mathcal{C})$, for $g'[\{\langle\frac{1}{2}, 0\rangle\}] = \{0\}$ generates $\mathfrak{C}$, because $\sim^{\mathfrak{A}} 0 = 1$. Then, since ε is an axiomatic equality determinant for both $\partial(\mathcal{A})$ and $\mathcal{C}$, by (3.1), we have $(\ker f') \subseteq (\ker g')$, in which case, by the Homomorphism Theorem, $h \triangleq (g' \circ f'^{-1}) \in \hom(\partial(\mathcal{A}), \mathcal{C})$, and so, since $D^{\mathcal{C}} = \{1\}$, we get $h(1) = 1$. Hence, $h(0) = h(\sim^{\mathfrak{A}} 1) = \sim^{\mathfrak{A}} h(1) = \sim^{\mathfrak{A}} 1 = 0$, while $1 = h(1) = h(\sim^{\mathfrak{A}}\frac{1}{2}) = \sim^{\mathfrak{A}} h(\frac{1}{2})$, in which case, as $h(\frac{1}{2}) \in 2$, $h(\frac{1}{2}) = \sim^{\mathfrak{A}}\sim^{\mathfrak{A}} h(\frac{1}{2}) = \sim^{\mathfrak{A}} 1 = 0$, and so $\hom(\mathfrak{A}, \mathfrak{A}) \ni h = h_0$.

- $\mathcal{A}$ is truth-singular,
 Then, by Lemmas 2.7 and 2.8, there are some set I, some submatrix $\mathcal{D}$ of $\mathcal{A}^I$, some Σ-matrix $\mathcal{E}$, some $g \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{D}, \mathcal{E})$ and some $f \in \hom_{\mathrm{S}}^{\mathrm{S}}(\partial(\mathcal{A}), \mathcal{E})$, in which case $\mathcal{E}$ is truth-singular, for $\mathcal{A}$ is so, and so $f(1) = f(\frac{1}{2})$. Hence, f is not injective, in which case, by Remark 2.5, $\partial(\mathcal{A})$ is not simple, and so, by Lemma 5.1, there are some $\sim$-classical Σ-matrix $\mathcal{B}$ and some $e \in \hom_{\mathrm{S}}^{\mathrm{S}}(\partial(\mathcal{A}), \mathcal{B})$. Therefore, by (2.13) and Theorem 6.5, there is some isomorphism i from $\mathcal{B}$ onto $\mathcal{A}{\restriction}2$, in which case $h \triangleq (i \circ e) \in \hom_{\mathrm{S}}(\partial(\mathcal{A}), \mathcal{A}{\restriction}2)$, and so $\hom(\mathfrak{A}, \mathfrak{A}) \ni h = h_1$. $\square$

Theorem 11.4. *Suppose both C is both $\bar{\wedge}$-conjunctive (viz., $\mathcal{A}$ is so) and not $\sim$-classical (in which case $\mathcal{A}$ is simple; cf. Theorem 5.3), and $\mathcal{A}$ is false-/truth-singular. Then, the following are equivalent:*

(i) *C is self-extensional;*
(ii) *$h_{0/(1|1-)} \in \mathrm{hom}(\mathfrak{A}, \mathfrak{A})$;*
(iii) *$\mathcal{A}_{1/(1+|0)} \in \mathrm{Mod}(C)$.*

Proof. First, (i)⇒(ii) is by Lemma 11.2. Next, (ii)⇒(iii) is by (2.13), (11.3) and (11.4). Finally, (iii)⇒(i) is by Theorem 3.9(vi)⇒(i), (11.1) and (11.2). □

First, by Theorem 6.5 and Lemma 11.2, we immediately have:

Corollary 11.5. *Suppose both C is both $\bar{\wedge}$-conjunctive (viz., $\mathcal{A}$ is so) and self-extensional, and $\mathcal{A}$ is false-singular (in particular, $\sim$-paraconsistent [viz., C is so]). Then, C is $\sim$-subclassical.*

Corollary 11.6. *Suppose C is both $\bar{\wedge}$-conjunctive, self-extensional and $\sim$-subclassical. Then, $\sim^{\mathfrak{A}}\frac{1}{2} \neq \frac{1}{2}$.*

Proof. By contradiction. For suppose $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$, in which case $(\sim^{\mathfrak{A}}\frac{1}{2} \in D^{\mathcal{A}}) \Leftrightarrow (\frac{1}{2} \in D^{\mathcal{A}})$, in which case $\mathcal{A}$ is not $\sim$-negative, and so, by Theorem 5.3, C is not $\sim$-classical. Hence, by Corollary 9.2 and Lemma 11.2, $h_{0/1} \in \mathrm{hom}(\mathfrak{A}, \mathfrak{A})$, whenever $\mathcal{A}$ is false-/truth-singular. Therefore, $(1/0) = \sim^{\mathfrak{A}}(0/1) = \sim^{\mathfrak{A}} h_{0/1}(\frac{1}{2}) = h_{0/1}(\sim^{\mathfrak{A}}\frac{1}{2}) = h_{0/1}(\frac{1}{2}) = (0/1)$. This contradiction completes the argument. □

Corollary 11.7. *Suppose $\mathcal{A}$ is both $\bar{\wedge}$-conjunctive (viz., C is so) and not $\sim$-negative, unless C is $\sim$-classical. Then, C is both self-extensional and $\sim$-subclassical iff both C has PWC with respect to $\sim$ and either C is $\sim$-classical or $\mathfrak{A}$ is a $\bar{\wedge}$-semilattice satisfying* (11.5).

Proof. First, assume C is both self-extensional and $\sim$-subclassical. Consider the following complementary cases:

- C is $\sim$-classical.
  Then, by Remark 2.6(i)**b)**, C has PWC with respect to $\sim$.
- C is not $\sim$-classical.
  Then, C is $\bar{\wedge}$-conjunctive, in which case, by Lemma 11.2 and Corollary 9.2, $\mathfrak{A}$ is a $\bar{\wedge}$-semilattice satisfying both (11.5) and $0 \leq^{\mathfrak{A}}_{\bar{\wedge}} \frac{1}{2}$, and so $\sim^{\mathfrak{A}}$ is anti-monotonic with respect to $\leq^{\mathfrak{A}}_{\bar{\wedge}}$. Hence, by Theorem 3.12(i)⇒(ii), C has PWC with respect to $\sim$.

Conversely, assume both C has PWC with respect to $\sim$ and either C is $\sim$-classical or $\mathfrak{A}$ is a $\bar{\wedge}$-semilattice satisfying (11.5). Consider the following complementary cases:

- C is $\sim$-classical.
  Then, it is, in particular, $\sim$-subclassical as well as self-extensional.
- C is not $\sim$-classical.
  Then, $\mathcal{A}$ is both $\bar{\wedge}$-conjunctive and non-$\sim$-negative as well as false-/truth-singular, in which case $\sim^{\mathfrak{A}}\frac{1}{2} \neq (0/1)$, and so $D^{\partial(\mathcal{A})} = (\sim^{\mathfrak{A}})^{-1}[A \setminus D^{\mathcal{A}}]$, while $\mathfrak{A}$ is a $\bar{\wedge}$-semilattice satisfying (11.5). Consider any $\phi \in \mathrm{Fm}^{\omega}_{\Sigma}$, any $\psi \in C(\phi)$, in which case $\sim\phi \in C(\sim\psi)$, and any $h \in \mathrm{hom}(\mathfrak{Fm}^{\omega}_{\Sigma})$ such that $h(\phi) \in D^{\partial(\mathcal{A})}$, in which case $h(\sim\phi) \notin D^{\mathcal{A}}$, and so $h(\sim\psi) \notin D^{\mathcal{A}}$, that is, $h(\psi) \in D^{\partial(\mathcal{A})}$. Thus, $\partial(\mathcal{A})$ is a $(2\setminus 1)$-model of C. In particular, it is weakly $\bar{\wedge}$-conjunctive, for C is so. Moreover, by (11.5) and the idempotencity identity for $\bar{\wedge}$ true in $\mathfrak{A}$, $D^{\partial(\mathcal{A})}$ is closed under $\bar{\wedge}^{\mathfrak{A}}$, in which case $\mathcal{A}_{1/+} = \partial(\mathcal{A})$ is $\bar{\wedge}$-conjunctive, and so, by Lemma 3.11, is a model of C. Hence,

by Theorem 11.4(iii)⇒(i), C is self-extensional. Finally, if it was not $\sim$-subclassical, then, by Lemma 11.2 and Corollary 9.2, $\mathcal{A}$ would be truth-singular, while h_{1-} would be an endomorphism of $\mathfrak{A}$, in which case, by (2.13) and (11.4), $\mathcal{A}_{0+}$ would be a model of C, and so the latter would not be $\overline{\wedge}$-conjunctive, for the former is not so, because of (11.5). □

11.1.1. *Both conjunctive and disjunctive logics.*

Corollary 11.8. *Suppose both C is both $\overline{\wedge}$-conjunctive and $\underline{\vee}$-disjunctive (viz., $\mathcal{A}$ is so; cf. Lemma 10.1), and both C is not $\sim$-classical and $\mathcal{A}$ is false-/truth-singular. Then, the following are equivalent:*

(i) *C is self-extensional;*
(ii) *$h_{0/1}$ is an endomorphism of $\mathfrak{A}$;*
(iii) *$\partial(\mathcal{A}) \in \mathrm{Mod}(C)$.*

Proof. First, assume (i) holds. Then, by Theorem 3.12(i)⇒(ii), $\mathfrak{A}$ is a $(\overline{\wedge}, \underline{\vee})$-lattice, in which case, as A is finite, $\flat^{\mathfrak{A}}_{\underline{\vee}}$ is the greatest element of the poset $\langle A, \leq^{\mathfrak{A}}_{\overline{\wedge}}\rangle$, while, as $1 \in D^{\mathcal{A}}$, whereas $\mathcal{A}$ is $\underline{\vee}$-disjunctive, we have $\flat^{\mathfrak{A}}_{\underline{\vee}} = (1 \underline{\vee}^{\mathfrak{A}} \flat^{\mathfrak{A}}_{\underline{\vee}}) \in D^{\mathcal{A}}$, and so, by Lemma 11.2(11.5), we get $\flat^{\mathfrak{A}}_{\underline{\vee}} = 1$. In particular, $0 \leq^{\mathfrak{A}}_{\overline{\wedge}} 1$. In this way, Lemma 11.2(iii)⇒(vii) yields (ii).

Next, (ii)⇒(iii) is by (2.13) and (11.3). Finally, (iii)⇒(i) is by Theorem 3.9(vi)⇒(i) and (11.1). □

This positively covers [the implication-less fragment of] Gödel's three-valued logic [2]. As for its negative instances, as a first one, we should like to highlight P^1, in which case $\mathfrak{A}$ has no semilattice (even merely idempotent and commutative) secondary operations, simply because the values of primary ones belong to 2. Likewise, three-valued expansions of HZ are not self-extensional, because, in that case, though $\mathcal{A}$, being false-singular, is neither $\wedge$-conjunctive nor $\vee$-disjunctive, simply because $\mathfrak{A}$ is a $(\wedge, \vee)$-lattice but with distinguished zero, $\mathfrak{A}$ is a $(\vee^{\sim}, \wedge^{\sim})$-lattice with zero 0 and unit $\frac{1}{2}$ — it is this *non-artificial* instance that warrants, in general, considering the case, when 1 is not a unit of the $(\overline{\wedge}, \underline{\vee})$-lattice $\mathfrak{A}$. As to more negative instances of Corollary 11.8, we need some its generic consequences.

First, as $(\mathrm{img}\, h_{0/1}) = 2$, by Theorem 6.5 and Corollary 11.8, we immediately have:

Corollary 11.9. *Suppose C is both $\overline{\wedge}$-conjunctive and $\underline{\vee}$-disjunctive (viz., $\mathcal{A}$ is so; cf. Lemma 10.1) as well as self-extensional. Then, C is $\sim$-subclassical.*

The condition of $(\mathcal{A}/C)$'s being false-singular/$\sim$-subclassical/$\underline{\vee}$-disjunctive can not be omitted in the formulation of Corollary 11.5/11.6/11.9, as it is demonstrated by:

Example 11.10. Let $\mathcal{A}$ be both canonical and truth-singular, $\Sigma = \{\wedge, \sim\}$, $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$ and

$$(a \wedge^{\mathfrak{A}} b) \triangleq \begin{cases} a & \text{if } a = b, \\ \frac{1}{2} & \text{otherwise,} \end{cases}$$

for all $a, b \in A$. Then, $\langle \sim^{\mathfrak{A}} 0, \sim^{\mathfrak{A}} \frac{1}{2}\rangle = \langle 1, \frac{1}{2}\rangle \notin \theta^{\mathcal{A}} \ni \langle 0, \frac{1}{2}\rangle$, in which case $\theta^{\mathcal{A}} \notin \mathrm{Con}(\mathfrak{A})$, while $(0 \wedge^{\mathfrak{A}} 1) = \frac{1}{2} \notin 2$, in which case 2 does not form a subalgebra of $\mathfrak{A}$, and so, by Theorem 5.3, C is not $\sim$-classical. On the other hand, $\mathcal{A}$ is $\overline{\wedge}$-conjunctive, while $h_{1-} \in \hom(\mathfrak{A}, \mathfrak{A})$, so by Theorem 11.4, C is self-extensional. In particular, by Corollary 11.6, C is not $\sim$-subclassical. □

First, by Corollaries 11.7 and 11.9, we immediately have:

Corollary 11.11. *Suppose $\mathcal{A}$ is both $\bar{\wedge}$-conjunctive and $\veebar$-disjunctive (viz., C is so; cf. Lemma 10.1) as well as not $\sim$-negative (in particular, either $\sim$-paraconsistent or $(\veebar, \sim)$-paracomplete [viz., C is so]), unless C is $\sim$-classical. Then, C is self-extensional iff both C has PWC with respect to $\sim$ and either C is $\sim$-classical or $\mathfrak{A}$ is a $\bar{\wedge}$-semilattice satisfying* (11.5).

Likewise, by Corollaries 11.6 and 11.9, we also get:

Corollary 11.12. *Suppose C is both $\bar{\wedge}$-conjunctive and $\veebar$-disjunctive (viz., $\mathcal{A}$ is so; cf. Lemma 10.1) as well as self-extensional. Then, $\sim^{\mathfrak{A}}\frac{1}{2} \neq \frac{1}{2}$.*

This negatively covers arbitrary three-valued expansions of Kleene's three-valued logic [4] (including Łukasiewicz' one $Ł_3$ [6]) and of LP (including LA) as well as of HZ. On the other hand, three-valued expansions of $Ł_3$, LA and HZ are equally covered by the next subsection.

11.2. **Implicative logics.**

Lemma 11.13. *Suppose $\mathcal{A}$ is both $\sqsupset$-implicative (and so $\veebar_{\sqsupset}$-disjunctive) and conjunctive (in particular, negative; cf. Remark 2.6(i)*a*)). Then, C is not self-extensional, unless it is $\sim$-classical.*

Proof. By contradiction. For suppose C is both self-extensional and non-$\sim$-classical. Then, by Corollary 11.8, $h_{0/1} \in \hom(\mathfrak{A}, \mathfrak{A})$, whenever $\mathcal{A}$ is false-/truth-singular, in which case $2 = (\mathrm{img}\, h_{0/1})$ forms a subalgebra of $\mathfrak{A}$, and so both $(\frac{1}{2} \sqsupset^{\mathfrak{A}} 0) = (0/1)$ and $((0/1) \sqsupset^{\mathfrak{A}} 0) = (1/0)$. Therefore, $(0/1) = h_{0/1}(0/1) = h_{0/1}(\frac{1}{2} \sqsupset^{\mathfrak{A}} 0) = (h_{0/1}(\frac{1}{2}) \sqsupset^{\mathfrak{A}} h_{0/1}(0)) = ((0/1) \sqsupset^{\mathfrak{A}} 0) = (1/0)$. This contradiction completes the argument. □

Corollary 11.14. *Suppose $\mathcal{A}$ is both truth-singular and $\sqsupset$-implicative. Then, C is not self-extensional, unless it is $\sim$-classical.*

Proof. Then, $(a \sqsupset^{\mathfrak{A}} a) = 1$, for all $a \in A$, in which case $\mathcal{A}$ is $\neg$-negative, where $(\neg x_0) \triangleq (x_0 \sqsupset \sim(x_0 \sqsupset x_0))$, and so Lemma 11.13 completes the argument. □

This immediately covers arbitrary three-valued expansions of $Ł_3$. The "false-singular" case is but more complicated. First, we have:

Corollary 11.15. *Suppose $\mathcal{A}$ is both false-singular and $\sqsupset$-implicative. Then, C is not self-extensional, unless it is either $\sim$-paraconsistent or $\sim$-classical.*

Proof. If C is not $\sim$-paraconsistent, then $\sim^{\mathfrak{A}}\frac{1}{2} = 0$, in which case $\mathcal{A}$ is $\sim$-negative, and so Lemma 11.13 completes the argument. □

Lemma 11.16. *Let C' be a Σ-logic, $\mathcal{B} \in \mathrm{Mod}^*(C')$, $a \in B$ and $\mathcal{D} \triangleq \langle \mathfrak{B}, \{a \sqsupset^{\mathfrak{B}} a\}\rangle$. Suppose C' is finitary, self-extensional and weakly $\sqsupset$-implicattive. Then, $\mathfrak{D} \in \mathrm{Mod}(C')$.*

Proof. Let $\varphi \in C'(\varnothing)$ and $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{B})$. Then, $V \triangleq \mathrm{Var}(\phi) \in \wp_{\omega}(V_{\omega})$. Take any $v \in (V_{\omega} \setminus V)$. Let $g \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{B})$ extend $(h \restriction V) \cup [v/a]$. Then, as, by (2.6), $(v \sqsupset v) \in C'(\varnothing)$, by Lemma 3.19, we have $h(\varphi) = g(\varphi) = g(v \sqsupset v) = (a \sqsupset^{\mathfrak{B}} a) \in D^{\mathcal{D}}$, and so $\mathcal{D} \in \mathrm{Mod}_1(C')$. Moreover, as, by (2.6), $(x_0 \sqsupset x_0) \in C'(\varnothing)$, by (2.7) and (2.8), we have $((x_0 \sqsupset x_0) \sqsupset x_1) \equiv^{\omega}_{C'} x_1$, in which case, by Corollary 3.8, we get $(a \sqsupset^{\mathfrak{B}} a) \sqsupset^{\mathfrak{B}} b) = b$, for all $b \in B$, and so (2.8) is true in $\mathcal{D}$. By induction on any $n \in \omega$, we prove that $\mathcal{D} \in \mathrm{Mod}_n(C')$. For consider any $X \in \wp_n(\mathrm{Fm}^{\omega}_{\Sigma})$, in which case $n \neq 0$, and any $\psi \in C(X)$. Then, in case $X = \varnothing$, we have $X \in \wp_1(\mathrm{Fm}^{\omega}_{\Sigma})$, and so $\psi \in \mathrm{Cn}_{\mathcal{D}}(X)$, for $\mathcal{D} \in \mathrm{Mod}_1(C')$. Otherwise, take any $\phi \in X$, in which case $Y \triangleq (X \setminus \{\phi\}) \in \wp_{n-1}(\mathrm{Fm}^{\omega}_{\Sigma})$, and so, by DT with respect to $\sim$, that C has, and the induction hypothesis, we have $(\phi \sqsupset \psi) \in C(Y) \subseteq \mathrm{Cn}_{\mathcal{D}}(Y)$. Therefore, by (2.8)$[x_0/\phi, x_1/\psi]$ true in $\mathcal{D}$, we eventually get $\psi \in \mathrm{Cn}_{\mathcal{D}}(Y \cup \{\phi\}) = \mathrm{Cn}_{\mathcal{D}}(X)$. Hence, since $\omega = (\bigcup \omega)$, we have $\mathcal{D} \in \mathrm{Mod}_\omega(C')$, and so $\mathcal{D} \in \mathrm{Mod}(C')$, for C' is finitary. □

Theorem 11.17. *Suppose $\mathcal{A}$ is both $\sqsupset$-implicative (viz., C is so; cf. Lemma 10.3), simple (i.e., C is not $\sim$-classical; cf. Lemma 5.1 and Corollary 10.2) and false-singular. Then, the following are equivalent:*

(i) *C is self-extensional;*
(ii) *$\mathcal{A}_{\frac{1}{2}} \in \mathrm{Mod}(C)$ is $\sim$-paraconsistent;*
(iii) *L_3 forms a subalgebra of $\mathfrak{A}^2$, $\mathfrak{A}^2 \restriction L_3$ being isomorphic to $\mathfrak{A}$;*
(iv) *$\sim^{\mathfrak{A}}$ is an automorphism of $\mathfrak{A}$;*
(v) *h_{1-} is an endomorphism of $\mathfrak{A}$;*
(vi) *$\mathcal{A}_{0+} \in \mathrm{Mod}(C)$.*

Proof. First, assume (i) holds. Then, by Corollary 11.15, C is $\sim$-paraconsistent, in which case $\sim^{\mathfrak{A}} \frac{1}{2} \neq 0$. Moreover, by (2.6), $a \triangleq (\frac{1}{2} \sqsupset^{\mathfrak{A}} \frac{1}{2}) \in D^{\mathcal{A}} = \{\frac{1}{2}, 1\}$. If a was not equal to $\frac{1}{2}$, then it would be equal to 1, and so would be $(b \sqsupset^{\mathfrak{A}} b)$, for any $b \in A$, in view of (2.6) and Lemma 3.19, in which case $\mathcal{A}$ would be $\neg$-negative, where $(\neg x_0) \triangleq (x_0 \sqsupset \sim(x_0 \sqsupset x_0))$, contrary to Lemma 11.13. Therefore, $a = \frac{1}{2}$. Hence, by Lemma 11.16, $\mathcal{A}_{\frac{1}{2}} \in \mathrm{Mod}(C)$. Moreover, by (2.6) and Lemma 3.19, $(b \sqsupset^{\mathfrak{A}} b) = \frac{1}{2}$, for all $b \in A$, in which case $\sim^{\mathfrak{A}}(b \sqsupset^{\mathfrak{A}} b) \in D^{\mathcal{A}}$, and so $\sim(x_0 \sqsupset x_0) \in C(\varnothing)$. Thus, by (2.6) and Lemma 3.19, $\sim^{\mathfrak{A}} a = a$, in which case $\mathcal{A}_{\frac{1}{2}}$ is $\sim$-paraconsistent, and so (ii) holds.

Next, assume (ii) holds, in which case, as $\mathcal{A}_{\frac{1}{2}}$ is truth-singular, by Theorem 7.3, L_3 forms a subalgebra of $\mathfrak{A}^2$, while $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$, and so $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$. Then, $\Upsilon \triangleq \{x_0, \sim x_0\}$ is a unary unitary equality determinant for $\mathcal{A}$, in which case, by the $\sqsupset$-implicativity of $\mathcal{A}$, $\{\phi \sqsupset \psi \mid (\phi \vdash \psi) \in \varepsilon_\Upsilon\}$ is an axiomatic binary equality determinant for $\mathcal{A}$, and so, by Lemmas 2.7, 2.8, 3.3, 3.4, 3.6 and Remark 2.5, there are some set I, some submatrix $\mathcal{B}$ of $\mathcal{A}^I$, and some $h \in \mathrm{hom}^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}_{\frac{1}{2}}, \mathcal{B})$. Let $a \triangleq h(\frac{1}{2})$ and $b \triangleq h(0)$, in which case $\sim^{\mathfrak{B}} a = h(\frac{1}{2})$ and $\sim^{\mathfrak{B}} b = h(1)$, and so $\{a/b, \sim^{\mathfrak{B}}(a/b)\} \subseteq (D^{\mathcal{B}}/(B \setminus D^{\mathcal{B}}))$. Hence, $a = (I \times \{\frac{1}{2}\})$ and $J \triangleq \{i \in I \mid \pi_i(b) = 0\} \neq \varnothing \neq K \triangleq \{i \in I \mid \pi_i(b) = 1\}$. Then, $e : A^2 \to A^I, \langle c, d\rangle \to ((J \times \{c\}) \cup (K \times \{d\}) \cup ((I \setminus (J \cup K)) \times \{\frac{1}{2}\}))$ is injective. Moreover, $e(\langle \frac{1}{2}, \frac{1}{2}\rangle) = a \in B$, and, for each $i \in 2$, $e(\langle i, 1-i\rangle) = (\sim^{\mathfrak{A}})^i b \in B$. Therefore, since $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$, $g \triangleq (e \restriction L_3)$ is an embedding of $\mathcal{D} \triangleq (\mathcal{A}^2 \restriction L_3)$ into $\mathcal{B}$, in which case $3 = |A| \leqslant |B| = |h[A]| \leqslant |A| = 3$, and so $|B| = 3$. In this way, h is injective, while $(\mathrm{img}\, g) = B$, in which case $g^{-1} \circ h$ is an isomorphism from $\mathcal{A}_{\frac{1}{2}}$ onto $\mathcal{D}$, and so from $\mathfrak{A}$ onto $\mathfrak{D}$. Thus, (iii) holds.

Further, assume (iii) holds, in which case $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$, and so $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$. Let e be any isomorphism from $\mathfrak{A}$ onto $\mathfrak{B} \triangleq (\mathfrak{A}^2 \restriction L_3)$. Then, as $\sim^{\mathfrak{B}} \langle i, 1-i\rangle = \langle 1-i, i\rangle \neq \langle i, 1-i\rangle$, for all $i \in 2$, we have $e(\frac{1}{2}) = \langle \frac{1}{2}, \frac{1}{2}\rangle$, in which case we get $e[2] = (2^2 \setminus \Delta_2)$, and so there is some $j \in 2$ such that $e(i) = \{\langle j, i\rangle, \langle 1-j, 1-i\rangle\}$, for each $i \in 2$. In this way, $\sim^{\mathfrak{A}} = (\pi_{1-j} \circ e) \in \mathrm{hom}(\mathfrak{A}, \mathfrak{A})$ is bijective. Thus, (iv) holds.

Now, assume (iv) holds. Then, $\sim^{\mathfrak{A}}[A/2] = (A/2)$, in which case $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$, and so $h_{1-} = \sim^{\mathfrak{A}} \in \mathrm{hom}(\mathfrak{A}, \mathfrak{A})$. Thus, (v) holds.

Furthermore, (v)⇒(vi) is by (2.13) and (11.4). Finally, (vi)⇒(i) is by (11.2) and Theorem 3.9(vi)⇒(i). □

First, by Theorems 7.3, 11.17 and Corollaries 11.14 and 11.15, we have the following refinement of the latter:

Corollary 11.18. *Suppose C is both $\sqsupset$-implicative (viz., $\mathcal{A}$ is so; cf. Lemma 10.3) and self-extensional. Then, it is non-maximally $\sim$-paraconsistent, unless it is $\sim$-classical.*

In particular, by Corollaries 9.6 and 11.18, we have the following minor refinement of Lemma 11.13:

Corollary 11.19. *Suppose C is both $\sqsupset$-implicative (viz., $\mathcal{A}$ is so; cf. Lemma 10.3) and self-extensional. Then, it is not weakly conjunctive, unless it is $\sim$-classical.*

Likewise, as opposed to Corollary 11.9, by Corollaries 10.9 and 11.18, we have:

Corollary 11.20. *Suppose C is both $\sqsupset$-implicative (viz., $\mathcal{A}$ is so; cf. Lemma 10.3) and self-extensional. Then, it is $\sim$-subclassical iff it is $\sim$-classical.*

Furthermore, as opposed to Corollary 11.12, we have:

Corollary 11.21. *Suppose C is both $\sqsupset$-implicative (viz., $\mathcal{A}$ is so; cf. Lemma 10.3) and self-extensional. Then, $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$.*

Proof. If $\sim^{\mathfrak{A}}\frac{1}{2}$ was not equal to $\frac{1}{2}$, then it would be equal to some $i \in 2$, in which case, since, by Theorem 11.17, $h_{1-} \in \hom(\mathfrak{A}, \mathfrak{A})$, we would have $(1-i) = h_{1-}(i) = h_{1-}(\sim^{\mathfrak{A}}\frac{1}{2}) = \sim^{\mathfrak{A}} h_{1-}(\frac{1}{2}) = \sim^{\mathfrak{A}}\frac{1}{2} = i$. □

Likewise, as opposed to Corollary 11.11, we have:

Corollary 11.22. *Suppose C is $\sqsupset$-implicative (viz., $\mathcal{A}$ is so; cf. Lemma 10.3). Then, it has PWC with respect to $\sim$ iff $\mathcal{A}$ is $\sim$-negative. In particular, any implicative $\sim$-paraconsistent/ both $\veebar$-disjunctive and $(\veebar, \sim)$-paracomplete Σ-logic with subclassical negation $\sim$ does not have PWC with respect to $\sim$.*

Proof. The "if" part is by Remark 2.6(i)**b)**. The converse is proved by contradiction. For suppose C has PWC with respect to $\sim$, and $\mathcal{A}$ is not $\sim$-negative. Without loss of generality, one can assume that $\sqsupset \in \Sigma$, in which case $\Sigma' \triangleq \{\sqsupset, \sim\} \subseteq \Sigma$, and so $\mathcal{A}' \triangleq (\mathcal{A}{\restriction}\Sigma')$ is both three-valued, $\sim$-super-classical, $\sqsupset$-implicative and non-$\sim$-negative as well as defines the Σ'-fragment C' of C. Then, C' is both $\sqsupset$-implicative and, by Remark 2.6(ii), Lemma 5.1 and Corollary 10.2, non-$\sim$-classical, for $\mathcal{A}$ is non-$\sim$-negative, as well as has PWC with respect to $\sim$. In particular, for any $\langle\phi, \psi\rangle \in \equiv^{\omega}_{C'}$ and any $\varphi \in \mathrm{Fm}^{\omega}_{\Sigma}$, we have both $\sim\phi \equiv^{\omega}_{C'} \sim\psi$, $(\phi \sqsupset \varphi) \equiv^{\omega}_{C'} (\psi \sqsupset \varphi)$ and $(\varphi \sqsupset \phi) \equiv^{\omega}_{C'} (\varphi \sqsupset \psi)$. Therefore, C' is self-extensional. Hence, by (2.6), Corollary 11.14 and Theorem 11.17(i)⇒(ii), both $x_0 \sqsupset x_0$ and $\sim(x_0 \sqsupset x_0)$ are theorems of C'. Then, we have $(x_0 \sqsupset x_0) \in C'(\varnothing) \subseteq C'(x_0)$, in which case, by PWC, we get $\sim x_0 \in C'(\sim(x_0 \sqsupset x_0)) \subseteq C'(\varnothing) \subseteq C'(x_0)$, and so, by (2.15) with $n = 1$ and $m = 0$, $\sim$ is not a subclassical negation for C'. In this way, Theorem 4.1/ and Lemma 10.1 complete the argument. □

Finally, existence of a self-extensional $\sqsupset$-implicative $\sim$-paraconsistent three-valued Σ-logic with subclassical negation $\sim$ is due to:

Example 11.23. Let $\mathcal{A}$ be both canonical and false-singular, $\Sigma \triangleq \{\supset, \sim\}$ with binary $\supset$, $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$ and

$$(a \supset^{\mathfrak{A}} b) \triangleq \begin{cases} \frac{1}{2} & \text{if } a = b, \\ b & \text{otherwise}, \end{cases}$$

for all $a, b \in A$. Then, $\mathcal{A}$ is both $\sqsupset$-implicative and $\sim$-paraconsistent, and so is C. And what is more, $h_{1-} \in \hom(\mathfrak{A}, \mathfrak{A})$, and so, by Theorem 11.17, C is self-extensional. □

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