Forbidding Edges Between Points in the Plane to Disconnect the Triangulation Flip Graph

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May 5, 2022

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Abstract

The ***flip graph*** for a set P of points in the plane has a vertex for every triangulation of P, and an edge when two triangulations differ by one ***flip*** that replaces one triangulation edge by another. The flip graph is known to be connected even if some triangulation edges are *constrained* to be used. We study connectivity of the flip graph when some triangulation edges are *forbidden*.

A set X of edges between points of P is a ***flip cut set*** if eliminating all triangulations that contain edges of X results in a disconnected flip graph. If X is a single edge it is called a ***flip cut edge***. The ***flip cut number*** of P is the minimum size of a flip cut set. We give an algorithm to test if an edge is a flip cut edge. For a set of n points in convex position (whose flip graph is the 1-skeleton of the associahedron) we prove that the flip cut number is $n - 3$.

2012 ACM Subject Classification Theory of computation → Computational geometry; Mathematics of computing → Combinatorial algorithms

Keywords and phrases triangulations, flip graph, reconfiguration, flip cut edge, flip cut number, associahedra, connectivity, convex polygon, algorithm, computational geometry, graph

1 Introduction

Given a set P of n points in the plane, which may include collinear points, an ***edge*** of P is a line segment pq that intersects P in exactly the two endpoints p and q. A ***triangulation*** of P is maximal set of non-crossing edges. Triangulations have important applications in graphics and mesh generation [2, 10] and are of significant mathematical interest [9].

A fundamental approach to understanding triangulations is by means of *flips*. A ***flip*** operates on a triangulation by removing one edge pq and adding another edge uv to obtain a new triangulation—of necessity, the edges pq and uv will cross and their four endpoints will form a convex quadrilateral with no other points of P inside it. For example, in Figure 1, edge a_1b_1 can be flipped to uv. In 1972, Lawson [12, 13] proved that any triangulation of point set P can be reconfigured to any other triangulation of P by a sequence of flips. This can be expressed as connectivity of the ***flip graph***, which has a vertex for every triangulation of P and an edge when two triangulations differ by a flip.

Although reconfiguring triangulations via flips is well studied [4], there are some very interesting open questions, and many properties of flip graphs remain to be discovered.

The case of points in convex position is especially interesting because there is a bijection between flips in triangulations of a convex point set and rotations in binary trees [18]. Finding the rotation distance between two binary trees is of great interest in biology for phylogenetic trees [8], and in data structures for splay trees [18]. Furthermore, the flip graph for n points in convex position is the 1-skeleton of an $(n - 3)$-dimensional polytope called the *associahedron* [14], or see [6]. See Figure 2. Although there is no geometric analogue of the

This is an abstract of a presentation given at CG:YRF 2022. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear in a conference with formal proceedings and/or in a journal.

Figure 1 The smallest point set that has a flip cut edge. The edge $e = uv$ is a flip cut edge since forbidding e leaves two possible triangulations (as shown) and neither one allows a flip.

associahedron for the case of triangulations of a general point set, some of its properties carry over to an abstract complex called the *flip complex*. For example, the 2-dimensional faces of the flip complex, like those of the associahedron, have size 4 or 5 [15].

An open frontier in the study of flip graphs has to do with expander properties, which would potentially lead to rapid mixing via random flips. For results on mixing in triangulations, see [5, 16, 17]. More generally, researchers study connectivity properties of flip graphs. Recently, Wagner and Welzl [19] showed that for n points in general position in the plane, the flip graph is $\lceil \frac{n}{2} - 2 \rceil$-connected. For points in convex position, the flip graph is $(n-3)$-connected, which follows from Balinski's theorem [1] applied to the 1-skeleton of the associahedron, see [19].

One intriguing thing about flip graphs of triangulations is that many properties carry over when we restrict to triangulations containing some specified non-crossing edges—so-called *constrained triangulations*. The subgraph of the flip graph consisting of triangulations that contain all the constrained edges is connected [7].

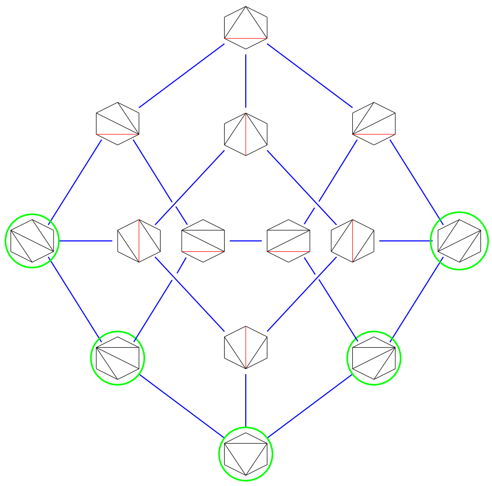

Figure 2 The flip graph of points of a convex hexagon is the 1-skeleton of an associahedron. If we forbid the two red edges, the resulting flip graph (with vertices circled in green) is connected.

Our Results. We study connectivity properties of the flip graph when—instead of constraining certain edges between points to be present—we *forbid* certain edges between points. To be precise, if a set X of edges between points is forbidden, we eliminate all triangulations that contain an edge of X, and examine whether the flip graph on the remaining triangulations is connected. We say that X is a ***flip cut set*** if the resulting flip graph is disconnected; in the special case where X is a single edge, we say that the edge is a ***flip cut edge***. For example the edge uv in Figure 1 is a flip cut edge, but the two red edges in Figure 2 do not form a flip cut set. Also see Figures 3, 4. We define the ***flip cut number*** of a set of points to be the minimum size of a flip cut set. This is analogous to the *connectivity* of a graph—the minimum number of vertices whose removal disconnects the graph.

Since the structure of the flip graph depends on the edges between the points, it seems more natural to study connectivity of the flip graph after deleting some of these edges, rather than deleting some vertices of the flip graph, as standard graph connectivity does, and as the result of Wagner and Welzl [19] does.

As our main result, we characterize when an edge e is a flip cut edge in terms of connectivity (in the usual graph sense) of the edges that cross e. We then use the characterization to give an $O(n \log n)$ time algorithm to test if a given edge e in a point set of size n is a flip cut edge. With that algorithm as preprocessing, we give a linear time algorithm to test if two triangulations are still connected after we eliminate from the flip graph all triangulations containing edge e.

For the case of n points in convex position, there are no flip cut edges and we show that the flip cut number is $n - 3$. For example, in Figure 2 the leftmost and rightmost triangulations become disconnected if we forbid one more edge, which yields a flip cut set of size 3 for $n = 6$.

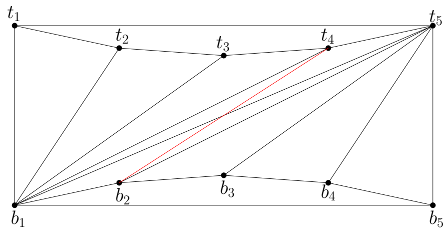

Figure 3 The "channel", and a triangulation that becomes frozen (an isolated vertex in the flip graph) if we forbid the edge b_2, t_{n-1} (in red). In fact, every edge $b_i t_j$, $i, j \notin \{1, 5\}$ is a flip cut edge.

We show that a point set of size n may have $\Theta(n^2)$ flip cut edges (see Figure 3), and we show that a flip cut edge may result in $\Theta(n)$ disconnected components in the flip graph. We also examine various special point sets whose flip graphs have been previously studied, such as points on an integer grid [5] and, more generally, point sets without empty convex pentagons [11]. Our characterization of flip cut edges becomes simpler in the absence of empty convex pentagons. Point sets without empty convex pentagons must have collinear points; our results do not assume points in general position.

For further details see the arxiv version [3].

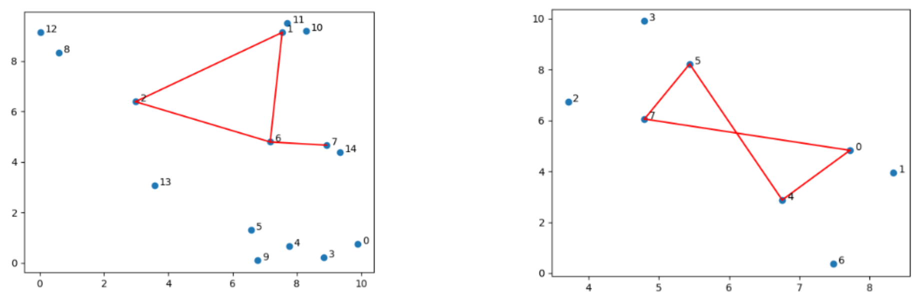

Figure 4 Some point sets and their flip cut edges (in red).

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