Maximum Cut Computation: Hopfield Neural Network

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ABSTRACT

In this research paper, the problem of computing the maximum cut in a graph is shown to be equivalent to finding the global minimum of energy function of the associated Hopfield Neural Network (HNN) . It is reasoned that using the initial condition based on the smallest eigenvector of synaptic weight matrix (i.e. eigenvector corresponding to smallest eigenvalue), in the serial mode of operation, HNN reaches the global minimum of associated energy function (quadratic form with the threshold vector being zero). Thus, maximum cut can be determined using HNN in serial mode of operation.

1. INTRODUCTION:

In an effort to model the biological neural network, Hopfield proposed an Artificial Neural Network (ANN) [5]. Such an ANN constitutes a homogeneous nonlinear dynamical system with the state vector lying on the symmetric unit hypercube (with no external input). Goles and Fogleman proved an interesting Theorem associated with the dynamics of Hopfield Associative memory (HAM). They associated an energy function with the state vector of the nonlinear dynamical system. It is related to a quadratic form associated with the symmetric synaptic weight matrix of Hopfield Neural Network (HNN) [5]. It is proved that HNN performs local optimization of the energy function (starting with the initial state and reaches the local/global peak of the energy hill).Bruck et.al proved that the state of HNN at which the global optimum of energy function is reached corresponds to the minimum cut in the undirected weighted graph based on the neuronal nodes as vertices and the synaptic connection based edges with the associated synaptic weights [7]. This result motivated the author to see if maximum cut in a weighted graph can be related to the dynamics of HNN. Culmination of those efforts resulted in this research paper.

This research paper is organized as follows. In Section 2, related research literature is reviewed. In Section 3, determination of maximum cut in a graph is related to the energy function of Hopfield Neural Network (HNN). The research paper concludes in Section 4.

2. REVIEW OF RELATED RESEARCH LITERATURE:

We now summarize the details of Hopfield Neural Network (HNN). HNN is an artificial neural network based on the McCulloch-Pitts model of neurons. Thus, every neuron is in the state { +1 or -1 }. Suppose, there are 'L' such neurons in the ANN. Let the state of the HNN at time index 'n' be captured by an 'L x 1' column vector, $\bar{\bar{V}}(n)$ $i.e.$ $\bar{V}(n) = [\, v_1(n) \;\; v_2(n) \ldots . v_L(n) \,] \;\; for\; n \geq 0.$

It is thus clear that the state vector lies on the corners of unit hypercube which are L dimensional vectors with { +1, -1 } components. The connection structure of the network of neurons consists of L vertices connected by edges with symmetric synaptic weights. There is a 'threshold value' at each neuronal node captured by an Lx1 vector $\bar{T}$. Thus, the synaptic weight matrix is a symmetric matrix, denoted by $\bar{K}$. Thus, the nonlinear, homogeneous dynamical system associated with HNN is initialized with the state vector (with {+1, -1 } components) and the state updation is performed according to the following two modes of operation [5] :

Serial Mode: The state updation at any time instant is performed at a single neuronal node 'i' in the following manner:

$$v_i(n+1) = Sign\left\{\sum_{j=1}^{L} K_{ij} v_j(n) - t_i\right\}, \; for \; n \geq 0$$

Fully Parallel Mode: The state updation at any time instant is performed at all the neuronal nodes (at once) i.e.

$$\bar{V}(n+1) = Sign\{\bar{K} \; \bar{V}(n) - \bar{T})\} \; for \; n \geq 0 ,$$

where $\bar{V}(n)$ is the state vector of HNN at time 'n'.

Other Parallel Modes: The state updation at any time instant is performed at more than one node, but strictly less than 'L' nodes (i.e. all nodes)

In the state space of HNN, there are distinguished states, called 'stable states' and 'anti-stable states'. They are defined in the following manner:

Stable State: A state vector $\bar{U}$ of HNN is called "stable state" if and only if

$$\bar{U} = Sign\{\bar{K} \; \bar{U} - \bar{T}\}.$$

i.e once the HNN state dynamics reaches a stable state, there will be no further change of state.

Anti-Stable State: A state vector $\bar{W}$ of HNN is called "anti-stable state" [4] if and only if

$$-\bar{W} = Sign\{\bar{K} \; \bar{W} - \bar{T}\}.$$

Hopfield Neural Network dynamics is captured through the following convergence Theorem:

Theorem 1: Starting in any initial state vector lying on the unit symmetric hypercube

(i) HNN dynamics converges to a stable state in the serial mode of operation, whereas

(ii) In the fully parallel mode of operation, the HNN dynamics converges to a stable state or cycle of length atmost 2.

Thus, the convergence Theorem confirms the operation of HNN as an associative memory.

3. MAXIMUM CUT DETERMINATION: HOPFIELD ASSOCIATIVE MEMORY:

The following concepts are associated with a graph and they naturally arises in the dynamics of HNN.

Consider an undirected, weighted graph with the set of vertices V and the set of edges E. Let $\hat{V}$ be a subset of vertex set V and let $\tilde{V}$ = V - $\hat{V}$ (i.e. $\tilde{V}$ contains vertices of graph that are not in $\widehat{V}$. The set of edges, each of which is incident at a vertex $\hat{V}$and at a vertex in $\tilde{V}$ is called a cut in the graph

- The weight of a cut is the sum of its edge weights
- A minimum cut of a graph is a cut with minimum weight.
- A maximum cut of a graph is a cut with maximum weight.

The proof of convergence Theorem is based on associating a quadratic energy function (e.g. quadratic form) with the dynamics of HNN (i.e. based on the state vector of HNN). It is reasoned that the energy function is non-decreasing in successive time instants (involving state updation). Thus, HNN performs local optimization of energy function. Based on this fact Bruck et.al proved the following Lemma which relates HAM dynamics with the determination of

minimum cut in the underlying undirected, weighted graph [7

Lemma 1: Consider a HNN with the synaptic weight matrix $\overline{K}$ and the threshold vector $\overline{T} \equiv \overline{0}$ (zero vector). The problem of finding a state V for which the energy is global maximum is equivalent to finding the minimum cut in the graph corresponding to HNN.

In [1],[2], [3] the following lemmas are proved in association with HNN with zero threshold vecto

Lemma 2: Let $\bar{Z}$ be a corner of unit hypercube which is an eigenvector of $\overline{K}$ corresponding to a positive eigenvalue μ. Then $\bar{Z}$ is a stable state of HNN with $\overline{K}$ as the synaptic weight matrix. i..

$$\bar{Z} = \quad Sign\{\overline{K}\ \bar{Z}\}$$

Lemma 3: Let $\bar{F}$ be a corner of unit hypercube which is an eigenvector of $\overline{K}$ corresponding to a negative eigenvalue μ. Then $\bar{F}$ is an anti-stable state of HNN with $\overline{K}$ as the synaptic weight matrix i.e

$$-\bar{F} = \quad Sign\{\overline{K}\ \bar{F}\}$$

The following Rayleigh's Theorem relates the eigenvalues, eigenvectors of a symmetric matrix $\overline{K}$ with the quadratic form associated with $\overline{K}$ evaluated at the vectors $\overline{U}$ lying on the unit Euclidean hypersphere i.e.

Rayleigh's Theorem: Let $\overline{U}^T\overline{K}\overline{U}$ be a quadratic form associated with symmetric matrix $\overline{K}$ evaluated at the vectors lying on the unit Euclidean hypersphere i.e $\overline{U}^T\overline{U} = 1$. The local/global optimum value of quadratic form ie. $\overline{U}^T\overline{K}\overline{U}$ occur at the eigenvectors of $\overline{K}$ with the corresponding quadratic form value being the eigenvalue.

Lemma 4: The problem of finding the global minimum value of energy function of HNN (based on $\overline{K}, with\ \overline{T}\ \equiv \overline{0}$) is equivalent to finding maximum cut in the graph corresponding to HNN.

Proof: Follows from similar argument for minimum cut in [7]. Details are avoided for brevity.

Thus, we have the following result:

(i) IF the eigenvector of $\overline{K}$ corresponding to smallest eigenvalue is a corner of hypercube, it corresponds to maximum cut in the graph associated with HNN (based on $\overline{K}$).

The following Theorem follows from the above results as well as the Theorem (6) in [6].

Theorem 2: Let $\bar{f}$ be the eigenvector of (HNN based on) synaptic weight matrix $\overline{K}$ corresponding to smallest eigenvalue (of $\overline{K}$). Let

$$\bar{P} = \quad Sign\{^{-}\bar{f}\}$$

be a corner of hypercube which is the initial condition vector of HNN (based on $\overline{K}$) running in serial mode of operation. The associated stable state (of $\overline{K}$) of HNN reached with such an initial condition $\bar{P}$ corresponds to minimum cut in the graph associated with $\overline{K}$.

Proof: Follows from proof of Theorem (6) in [6] for the case of smallest eigenvalue

4. CONCLUSION:

In this research paper, the problem of determining the maximum cut is related to determining the global minimum of quadratic energy function of the associated Hopfield Neural Network. An approach to find such global minimum is proposed based on the eigenvector of synaptic weight matrix (of HNN) corresponding to the smallest eigenvalue.

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